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CROSSED PRODUCTS OF CONTINUOUS-TRACE C^*-ALGEBRAS BY SMOOTH ACTIONS

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ABSTRACT. We study in detail the structure of C^*-crossed products of the form $A \rtimes_\alpha G$, where A is a continuous-trace algebra and α is an action of a locally compact abelian group G on A, especially in the case where the action of G on $\hat{A}$ has a Hausdorff quotient and only one orbit type. Under mild conditions, the crossed product has continuous trace, and we are often able to compute its spectrum and Dixmier-Douady class. The formulae for these are remarkably interesting even when G is the real line.

In recent years, considerable progress has been made in understanding the structure of a transformation group C^*-algebra $C^*(G, X)$ (sometimes written $C_0(X) \rtimes G$) when the orbit space X/G is reasonable. For example, a well-known theorem by Green [**10**] asserts that if G acts freely and properly on X, then $C^*(G, X)$ is isomorphic to $C_0(X/G, \mathcal{K}(L^2(G)))$ (for a good discussion of this, and some generalizations, see [**29**]), and for abelian G a description of the topology on the spectrum of $C^*(G, X)$ has been given by Williams [**38**]. The structure of crossed products $A \rtimes_\alpha G$ with A noncommutative is much more complicated, due to an extra step in the description of their representation theory by the "Mackey machine" (e.g., [**34** and **11**]): if $\pi \in \hat{A}$, there is an obstruction in $H^2(G_\pi, \mathbb{T})$ to extending π to a covariant representation of (A, G_π), where G_π is the stabilizer of π in G, and even if this obstruction vanishes, it may not be possible to find a *canonical* extension of π. Here we shall study the crossed product $A \rtimes_\alpha G$, when A is a continuous-trace C^*-algebra, G is abelian, and the action of G on the spectrum X of A satisfies various local triviality hypotheses. Somewhat surprisingly, our point of view yields new information even about transformation group algebras. For example, it turns out that for a certain action of $\mathbb{R}$ on S^3, the associated transformation group algebra has nonzero Dixmier-Douady invariant (Example 4.6 below).

There are two extreme cases which have already been investigated to some extent. First of all, *locally unitary* automorphism groups α: $G \to \operatorname{Aut} A$ [**27**] are group actions which act trivially on the spectrum and for which Mackey obstructions do not arise, and they include *all* such actions if G and A are separable and G is

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compactly generated and abelian [**32**, §2]. For such α, the spectrum of the crossed product $A \rtimes_\alpha G$ is in a natural way a principal $\hat{G}$-bundle over $\hat{A}$ [**27**, Theorem 2.2], and in fact $A \rtimes_\alpha G$ is isomorphic to the balanced tensor product

$$C_0\big((A \rtimes_\alpha G)^\wedge\big) \otimes_{C_0(X)} A$$

[**28**, Proposition 1.5].

Secondly, some things are known about *diagonal* actions on the pull-backs of C^*-algebras. If B is a C^*-algebra with spectrum T and $p\colon \Omega \to T$ is a principal G-bundle, then the pull-back p^*B is by definition $C_0(\Omega) \otimes_{C_0(T)} B$ (so the previously quoted result asserts $A \rtimes_\alpha G \cong p^*A$ for $p\colon (A \rtimes_\alpha G)^\wedge \to A$ when α is locally unitary). A diagonal action on p^*B, denoted $p^*\beta$, is one inherited from a tensor product action $\gamma \otimes \beta$ of G on $C_0(\Omega) \otimes B$, where γ is translation on Ω and β is an action of G on B which commutes with the action of $C_b(T) \cong Z(M(B))$. The spectrum of p^*B is canonically homeomorphic to Ω, and $p^*\beta$ induces the original action of G on Ω, so these give nontrivial examples of actions which induce principal bundle structures on the spectrum. In [**28**] it is shown that $p^*B \rtimes_{p^*\beta} G$ is often, but not always, Morita equivalent to B. (It is if β is implemented by a unitary group, or if the bundle $p\colon \Omega \to T$ is trivial.)

We begin where [**28**] leaves off: we consider actions of G on A which make $\hat{A}$ into a locally trivial principal G-bundle over $\hat{A}/G$. Our first main result (Theorem 1.1) says that, up to stable isomorphism, the pull-back actions studied in [**28**] are the only examples of such automorphism groups. The crossed product is again a continuous-trace C^*-algebra, and we give a formula for its Dixmier-Douady class

$$\delta\big(p^*B \rtimes_{p^*\beta} G\big)$$

in the case where β is locally unitary: the ingredients in our formula are $\delta(B)$, the class of the bundle $p\colon \Omega \to T$, and the class of the $\hat{G}$-bundle $(B \rtimes_\beta G)^\wedge \to \hat{B}$ constructed in [**27**].

Next we consider automorphism groups $\alpha\colon G \to \operatorname{Aut} A$ such that the action of G on $\Omega = \hat{A}$ has a fixed isotropy group H, and such that the induced action of G/H gives Ω the structure of a locally trivial principal G/H-bundle over a space $T = \Omega/G$. When the restriction of α to H is locally unitary (recall this is automatic if $H = \mathbb{Z}$, $\mathbb{T}$, or $\mathbb{R}$), the spectrum of $A \rtimes_{\alpha|_H} H$ is a principal $\hat{H}$-bundle over Ω, and it turns out that we have a commutative diagram of locally trivial principal bundles

$$\begin{array}{ccccc} & & (A \rtimes_{\alpha|_H} H)^\wedge & & \\ & {\scriptstyle \pi^*p}\swarrow & & \searrow{\scriptstyle p^*\pi} & \\ (A \rtimes_\alpha G)^\wedge & & & & \Omega, \\ & {\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle p} & \\ & & T & & \end{array}$$

where the SE arrows are $\hat{H}$-bundles and the SW arrows are G/H-bundles. The proof of this uses Green's version of the Mackey machine for twisted crossed products, and the realization of $A \rtimes_\alpha G$ as a twisted crossed product of $A \rtimes_{\alpha|_H} H$ by G/H.

In fact, $A \rtimes_\alpha G$ again has continuous trace, and by duality its Dixmier-Douady class must pull back under π^*p to $\delta(A \rtimes_{\alpha|_H} H) = (p^*\pi)^*\delta(A)$.

Our motivating example for this analysis is the case $G = \mathbb{R}$. The only nontrivial closed subgroups of $\mathbb{R}$ are isomorphic to $\mathbb{Z}$, so actions with reasonable orbit space and one orbit type fall into 3 classes: trivial, free and periodic. The first was discussed in **[27]**, we have already handled the second, and it remains to analyze the periodic case. The restriction of α to the isotropy group $\mathbb{Z}$ is automatically locally unitary, and $\Omega \to T$ is a principal bundle for the free action of the quotient $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by a theorem of Gleason **[6]**. We therefore obtain a commutative diamond of principal $\mathbb{T}$-bundles.

In fact $\mathbb{R}$-actions turn out to be particularly interesting, and we can say substantially more in this case. First of all, we prove a vanishing theorem for the Moore cohomology group $H^2(\mathbb{R}, C(X, \mathbb{T}))$, where X is any $\mathbb{R}$-space (satisfying a mild technical condition). As well as answering a question left open in **[14]**, this implies that any action of $\mathbb{R}$ on a continuous-trace algebra A is determined up to exterior equivalence by its action on the spectrum X of A. Secondly, it turns out that if A is stable (modulo a technical hypothesis on X) then every free action of $\mathbb{T}$ on X lifts to an action α of $\mathbb{R}$ on A. To see that this is not completely obvious, note that it cannot lift to an action of $\mathbb{T}$ unless the algebra A is a pull-back along the bundle $X \to T = X/\mathbb{T}$. These two results together imply that the action α, and hence also π and $\delta(A \rtimes_\alpha \mathbb{R})$, are determined uniquely by $[p] \in H^2(T, \mathbb{Z})$ and $\delta(A) \in H^3(X, \mathbb{Z})$. We show that these invariants are related by the Gysin sequences of the bundles π and p.

We have organized our work as follows. We begin with a §0 on preliminary matters, where we set up notation and review material from **[25, 26, 27, 28,** and **14]**. In fact, our presentation includes some minor improvements and new observations. We then discuss our results on automorphism groups which act freely on the spectrum. Our realization of these as pull-backs is Theorem 1.1, and our formula for the Dixmier-Douady class of the crossed product is Theorem 1.5. We also apply this formula to the problem of realizing a given automorphism group of p^*B as a pull-back of a locally unitary group on B.

In §2 we study the spectrum of $A \rtimes_\alpha G$ when the action of G on $\Omega = \hat{A}$ has fixed isotropy group H. The commutative diamond is valid in considerable generality (Proposition 2.1) and consists of principal bundles when $\alpha|_H$ is locally unitary and $\Omega \to \Omega/G$ locally trivial for the action of G/H (Theorem 2.2). In our §3 we present some examples satisfying these hypotheses. The first class of examples consists of induced C^*-algebras—given an action of a subgroup H of G on a C^*-algebra D, there is an action of G on a C^*-algebra $\operatorname{Ind}_H^G D$ of sections of an induced bundle of C^*-algebras over G/H. For the induced action we can identify explicitly the various topological invariants associated to our diamond. The second set of examples involves actions on the twisted transformation group C^*-algebras $C^*(G, X, \omega)$ of Wassermann (**[35**, §1]; see also **[28**, §4]). For these all the invariants can be nontrivial simultaneously. We also show how to construct explicitly actions of $\mathbb{R}$ on algebras A with spectrum S^3 for which the induced action of $\mathbb{R}/\mathbb{Z}$ on $\hat{A}$ is the Hopf fibration.

Our final section contains a detailed study of actions of $\mathbb{R}$. The results of §§2 and 3 already indicate that the spectrum of $A \rtimes_\alpha \mathbb{R}$ can be more complicated than previously expected: the conjecture of [**31**] would have implied that as a $\mathbb{T}$-bundle $(A \rtimes_\alpha \mathbb{R})^\wedge$ was always trivial. However, our cohomology vanishing theorem (4.1) enables us to verify this conjecture when $\delta(A) = 0$ (Theorem 4.8), which case seems the most likely to arise in the applications envisaged in [**31**]. In fact, we show the system $(A \otimes \mathscr{K}, \alpha \otimes \mathrm{id})$ is equivalent to $(C_0(X) \otimes \mathscr{K}, \tau \otimes \mathrm{id})$, where τ is translation on X, and the conjecture reduced to Williams' theorem [**38**, Theorem 5.3]. In Theorem 4.10, we prove that an action of $\mathbb{R}$ on a manifold M can be lifted to any stable continuous-trace algebra over M, and we note as a corollary that this makes it possible to construct some new examples of simple C^*-algebras. Our results on lifting circle actions on $\hat{A}$ to actions of $\mathbb{R}$ on A, and the associated topological formulae, are in Theorem 4.12. (The proofs of Theorems 4.10 and 4.12 use a certain amount of machinery from differential geometry, differential topology, and homotopy theory.) Then we continue with brief discussions of how our results assist the calculation of $A \rtimes_\alpha \mathbb{R}$, when the action of $\mathbb{R}$ on $\hat{A}$ has more than one orbit type, and in particular, the calculation of the C^*-algebra of a solvable Lie group. We consider the question of when the Dixmier-Douady invariant of continuous-trace subquotients of such an algebra can be nonzero, and conclude with a discussion of what one expects for exponential solvable Lie groups.

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0. Preliminaries. In this section, we review a number of basic definitions and concepts that will be used later, and fix a number of notations. We include a few results of a preliminary nature, the most important of which (Theorems 0.8 and 0.11 below) should be viewed as supplements to [**26** and **14**], respectively.

Automorphisms and automorphism groups of C^*-algebras. An *automorphism* of a C^*-algebra will always mean a $*$-isomorphism (necessarily isometric). If A is a C^*-algebra, $\mathrm{Aut}(A)$ will denote the group of all automorphisms of A. This is a topological group with respect to the topology of *pointwise* convergence. (We shall never use the norm topology on $\mathrm{Aut}(A)$.) When A is separable, $\mathrm{Aut}(A)$ is a Polish group, i.e., may be given a complete separable metric. (For equivalent properties, see [**20**, §2].)

An automorphism $\alpha \in \mathrm{Aut}(A)$ is called *inner* if there is some unitary element u in the multiplier algebra $M(A)$ such that $\alpha = \mathrm{Ad}\, u$, i.e., $\alpha(a) = uau^*$ for all $a \in A$. The inner automorphisms of A constitute a normal subgroup $\mathrm{Inn}(A)$ of $\mathrm{Aut}(A)$ which is usually *not* closed in $\mathrm{Aut}(A)$. However $\mathrm{Inn}(A)$ is naturally isomorphic to the quotient of $U(M(A))$, the unitary group of $M(A)$ equipped with the strict topology (see [**24**, §3.12]), by its center. In fact, the center of $U(M(A))$ may by the Dauns-Hofmann Theorem [**24**, §4.4] be identified with $C(\mathrm{Prim}\, A, \mathbb{T})$, where $\mathbb{T}$ is the circle and $\mathrm{Prim}\, A$ is the primitive ideal space of A, equipped with the hull-kernel topology. Here one may replace $\mathrm{Prim}\, A$ by its largest Hausdorff quotient (see [**1**, p.

268]), and the strict topology corresponds to uniform convergence on compacta. The following lemma compensates in part for the fact that $\mathrm{Inn}(A)$ may not be closed in $\mathrm{Aut}(A)$.

LEMMA 0.1. *Let A be a separable C^*-algebra. Then* $\mathrm{Inn}(A)$ *is a Borel subset of* $\mathrm{Aut}(A)$, *and the relative Borel structure on it coincides with that given by the identification with the Polish group* $U(M(A))/C(\mathrm{Prim}\,A, \mathbb{T})$.

PROOF. The map

$$\mathrm{Ad}\colon U(M(A)) \to \mathrm{Aut}(A)$$

is clearly a continuous homomorphism with kernel $C(\mathrm{Prim}\,A, \mathbb{T})$, hence gives a continuous embedding of $U(M(A))/C(\mathrm{Prim}\,A, \mathbb{T})$ into $\mathrm{Aut}(A)$, with image $\mathrm{Inn}(A)$. When A is separable, $\mathrm{Aut}(A)$ and $U(M(A))$ are Polish groups, hence $\mathrm{Inn}(A)$ is Borel and the embedding is a Borel isomorphism by [**19**, Theorem 3.2]. □

COROLLARY 0.2. *Suppose A is separable C^*-algebra, G is a Polish group, and φ: $G \to$* $\mathrm{Aut}(A)$ *is a continuous homomorphism or crossed homomorphism that happens to have image contained in* $\mathrm{Inn}(A)$. *Then φ is actually continuous for the Polish topology on* $\mathrm{Inn}(A)$ *obtained from the identification with* $U(M(A))/C(\mathrm{Prim}\,A, \mathbb{T})$.

PROOF. Since φ is continuous, it is certainly Borel, hence by the lemma is Borel with respect to the Polish topology on $\mathrm{Inn}(A)$. But then φ is continuous by an automatic continuity theorem of Banach (see for example [**20**, Proposition 5]). □

Now a *group action* on A, often called an *automorphism group* G of A (or *C^*-dynamical system*) means a continuous homomorphism from a topological group G (usually locally compact) into $\mathrm{Aut}(A)$. A group action α: $G \to \mathrm{Aut}(A)$ is called *inner* or *unitary* if it is implemented by a homomorphism u: $G \to U(M(A))$, continuous with respect to the strict topology on $M(A)$. (In other words, $\alpha_g(a) = u_g a u_g^*$ for all $g \in G$ and $a \in A$, and $g \mapsto u_g a$, $g \mapsto a u_g$ are continuous for any fixed $a \in A$.) When $G = \mathbb{Z}$, this notion coincides with that of innerness of the single automorphism $\alpha(1) = \alpha_1$. A weaker notion is for a group action to be *locally unitary*, which means that for each point in $\hat{A}$, one can choose a map u so that the condition $\alpha_g(a) = u_g a u_g^*$ holds *locally* in a neighborhood of this point [**27**].

Cohomology. We shall need throughout this paper to refer to cohomology groups of both topological spaces and topological groups. In the case of spaces, we shall work exclusively with sheaf cohomology, which coincides with the Čech theory since our spaces will be paracompact. The following lemma will be used later.

LEMMA 0.3. *Let X be a compact metric space, or more generally, a space with the homotopy type of a compact metric space. Then the* (*Čech*) *cohomology groups* $H^n(X, \mathbb{Z})$ *are countable.*

PROOF. For any paracompact space X, to compute $H^*(X, \mathbb{Z})$ it is enough to compute $H^*(\mathcal{U}, \mathbb{Z})$ as $\mathcal{U}$ runs over a cofinal family of open coverings of X. When X is compact, one may take each covering $\mathcal{U}$ to be finite, and when X is second-countable, a countable number of $\mathcal{U}$'s suffice. Now for $\mathcal{U}$ finite, the Čech cochain groups $C^n(\mathscr{U}, \mathbb{Z})$ are obviously finitely generated, hence so are the groups $H^n(\mathscr{U}, \mathbb{Z})$. So for X compact metric, $H^n(X, \mathbb{Z})$ is a countable direct limit of finitely generated groups, and so is countable. Since Čech cohomology with discrete coefficients is homotopy-invariant, it actually suffices for X to have the homotopy type of a compact metric space. □

REMARK 0.4. There are many quite nice second-countable locally compact spaces for which the Čech cohomology groups are uncountable. For instance, if X is an infinite disjoint union of copies of S^n, $H^n(X, \mathbb{Z})$ is an infinite *product* (not sum!) of copies of $\mathbb{Z}$, and so is uncountable. If $n \geqslant 1$, this pathology can arise even when X is a connected open manifold (such as a surface of infinite genus).

In fact, there are natural ways to topologize $H^n(X, \mathbb{Z})$. When X is compact, this group will be discrete, but for X noncompact it may fail to be Hausdorff. For instance, if $X = \varinjlim X_k$ is a limit of finite complexes X_k, then $H^n(X, \mathbb{Z})$ will contain $\varprojlim^1 H^{n-1}(X_k, \mathbb{Z})$ [**36**, pp. 272–273], which sometimes is a non-Hausdorff quotient of $\prod_k H^{n-1}(X_k, \mathbb{Z})$. □

For A a topological abelian group, we shall denote by $\underset{\sim}{A}$ the sheaf of germs of continuous A-valued functions over some (implicit) space. When A is discrete, $\underset{\sim}{A}$ may be identified with the constant sheaf A. Recall that for X paracompact, $H^1(X, \underset{\sim}{A})$ classifies locally trivial principal A-bundles. Thus $H^1(X, \underset{\sim}{\mathbb{T}}) \cong H^2(X, \mathbb{Z})$ classifies principal $\mathbb{T}$-bundles over X. If X is locally compact and paracompact, the Dixmier-Douady invariant of a continuous-trace algebra with spectrum X lives in $H^2(X, \underset{\sim}{\mathbb{T}}) \cong H^3(X, \mathbb{Z})$.

The sort of group cohomology that will be relevant for us is the so-called "Borel cochain" theory of C. C. Moore [**20**]. This theory associates groups $H^n(G, A)$ to a pair G, A, where G is a second-countable locally compact group and A is a Polish G-module. These groups are often the natural setting for certain obstructions to triviality of group actions on operator algebras (see, for instance, [**21**, **14**, and **32**]).

Continuous-trace algebras and their automorphisms. Recall that a C^*-algebra A is said to have *continuous trace* if $\hat{A}$ is Hausdorff and

$$\{a \in A_+ \colon \pi \mapsto \operatorname{tr} \pi(a) \text{ is finite-valued and continuous on } \hat{A}\}$$

is dense in the positive cone A_+ of A. The most basic such algebra is $\mathscr{K}(\mathscr{H})$, the algebra of compact operators on a Hilbert space $\mathscr{H}$. When $\mathscr{H}$ is separable and infinite-dimensional, we often write $\mathscr{K}$ for $\mathscr{K}(\mathscr{H})$. A C^*-algebra A is called *stable* if $A \otimes \mathscr{K} \cong A$. A stable separable continuous-trace algebra A with fixed spectrum X is classified up to $C_0(X)$-isomorphism by its *Dixmier-Douady invariant* $\delta(A) \in H^3(X, \mathbb{Z})$, and all classes in H^3 can arise. We shall use the following facts about automorphisms of continuous-trace algebras.

THEOREM 0.5 [**26**]. *Let A be a continuous-trace algebra with paracompact spectrum X and Dixmier-Douady invariant δ.*

(a) $\operatorname{Aut}_{C_0(X)} A$, *the subgroup of* $\operatorname{Aut}(A)$ *consisting of automorphisms leaving the spectrum pointwise fixed, coincides with the group of locally inner automorphisms.*

(b) *There is a short exact sequence*

$$1 \to \mathrm{Inn}(A) \to \mathrm{Aut}_{C_0(X)} A \xrightarrow{\zeta} H^2(X, \mathbb{Z}),$$

and if A is stable, ζ is surjective.

(c) *There is a short exact sequence*

$$1 \to \mathrm{Aut}_{C_0(X)} A \to \mathrm{Aut}(A) \xrightarrow{\rho} \mathrm{Homeo}_\delta(X),$$

where $\mathrm{Homeo}_\delta(X)$ *denotes the subgroup of the homeomorphism group of X that preserves δ. If A is separable and stable, ρ is surjective.* □

REMARK 0.6. It is obvious that $\mathrm{Aut}_{C_0(X)} A$ is closed in $\mathrm{Aut}(A)$ and that the map ρ: $\mathrm{Aut}(A) \to \mathrm{Homeo}(X)$ is continuous (for the compact-open topology on homeomorphisms). When $\delta = 0$, the exact sequence of (c) splits; however, we shall see that this is not usually the case otherwise. □

COROLLARY 0.7. *If A is a separable stable C^*-algebra with spectrum X and Dixmier-Douady invariant δ, there is a short exact sequence*

$$1 \to H^2(X, \mathbb{Z}) \to \mathrm{Out}(A) \to \mathrm{Homeo}_\delta(X) \to 1,$$

where $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$. □

THEOREM 0.8. *Let A be a separable C^*-algebra with spectrum X such that $H^2(X, \mathbb{Z})$ is countable.* (*By Lemma* 0.3, *this is automatic if X is compact or homotopy-equivalent to a compact space.*) *Then* $\mathrm{Inn}(A)$ *is open in* $\mathrm{Aut}_{C_0(X)} A$ *and closed in* $\mathrm{Aut}(A)$.

PROOF. When A is separable, $\mathrm{Aut}_{C_0(X)} A$ is a closed subgroup of the Polish group $\mathrm{Aut}(A)$, hence is Polish. By Lemma 0.1, $\mathrm{Inn}(A)$ is a Borel subgroup, and by hypothesis together with Theorem 0.5(b), it has countable index. Thus if $\mathrm{Aut}_{C_0(X)} A/\mathrm{Inn}(A)$ is given the discrete topology, the quotient map $\mathrm{Aut}_{C_0(X)} A \to \mathrm{Aut}_{C_0(X)} A/\mathrm{Inn}(A)$ is a Borel homomorphism from one Polish group onto another, hence is continuous and open. (See also [**21**, Proposition 6].) □

REMARK 0.9. The nature of the proof of Theorem 0.8 suggests that if A is a continuous-trace algebra with $H^2(\hat{A}, \mathbb{Z})$ uncountable, then perhaps $\mathrm{Inn}(A)$ might not be closed in $\mathrm{Aut}(A)$. In fact this can occur even when A is separable, as the following example illustrates.

Let X be the infinite mapping cylinder constructed in [**36**, p. 272], from iterations of a map $S^1 \to S^1$ of degree 2. The space X is an infinite, locally finite CW complex, hence is locally compact but not compact. By construction, X is a union of finite skeleta X_n with $H^1(X_n, \mathbb{Z}) \cong \mathbb{Z}$, $H^2(X_n, \mathbb{Z}) = 0$, and with the maps $H^1(X_{n+1}, \mathbb{Z}) \to H^1(X_n, \mathbb{Z})$ equal to multiplication by 2. By construction,

$$H^2(X, \mathbb{Z}) \cong \varprojlim{}^1\left(\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z}\right)$$

is uncountable. Let $A = C_0(X, \mathcal{K}(\mathcal{H}))$, when $\mathcal{H}$ is a separable infinite-dimensional Hilbert space. Then

$$\mathrm{Aut}_{C_0(X)} A \cong C(X, PU(\mathcal{H})),$$

where the maps from X to the projective unitary group PU (which is a $K(\mathbb{Z}, 2)$-space) are given the compact-open topology. The map $\mathrm{Aut}_{C_0(X)} A \to H^2(X, \mathbb{Z})$ of Theorem 0.5 is exactly the map sending a function v: $X \to PU$ to its homotopy class in $[X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z})$. Since $H^2(X_n, \mathbb{Z}) = 0$ for all n, any such ν is null-homotopic when restricted to any X_n, and thus ν may be approximated uniformly on compacta by null-homotopic maps (which correspond to inner automorphisms). This shows Inn A is dense in $\mathrm{Aut}_{C_0(X)} A$ even though the quotient $H^2(X, \mathbb{Z})$ is nontrivial (in fact, uncountable).

We take this opportunity to correct what seems to be an erroneous remark on p. 122 of [5]. While it is true that continuous-trace algebras have all their central sequences trivial, it seems to us that the argument used by Connes to prove equivalence of this condition with triviality of $\overline{\mathrm{Inn}}\, A/\mathrm{Inn}\, A$, when translated over to the C^*-context, breaks down precisely in the case of nonunital algebras with $\hat{A}$ noncompact. This explains why there can be a separable continuous-trace algebra with $\overline{\mathrm{Inn}} A/\mathrm{Inn}\, A$ nontrivial. □

Notions of equivalence for group actions. If α, β: $G \to \mathrm{Aut}(A)$ are two actions of a locally compact group G on a C^*-algebra, there are several plausible candidates for "equivalence" of the two actions. One is that they be related by an automorphism $\gamma \in \mathrm{Aut}(A)$, i.e., that $\alpha_g = \gamma\beta_g\gamma^{-1}$ for all $g \in G$. The notion of equivalence may be made more restrictive if one requires γ to fix $\hat{A}$ or Prim(A) pointwise, to be locally inner, or even inner. Another useful notion is that of *exterior equivalence*; α and β are exterior equivalent if they differ by a cocycle $G \to U(M(A))$ or if they may be chosen to be opposite corners of an action of G on $M_2(A)$ (2-by-2 matrices over A) [**24**, Lemma 8.11.2]. Exterior equivalence may be combined with equivalence via an isomorphism, i.e., one might say α and β are equivalent if α is exterior equivalent to $g \mapsto \gamma\beta_g\gamma^{-1}$ for some γ (again, various restrictions on γ are possible). The weakest reasonable notion of equivalence is for α and β to be called equivalent if the C^*-crossed products, which we shall denote $A \rtimes_\alpha G$ and $A \rtimes_\beta G$, are isomorphic. Usually, however, one would also like to keep track of the dual coaction of $\hat{G}$. The following theorem is due to G. K. Pedersen [**25**, (35)] however, as the statement there is not completely precise, we restate it for convenience.

THEOREM 0.10. *Let G be a locally compact abelian group and let α, β: $G \to \mathrm{Aut}(A)$ be two actions of G on a C^*-algebra. Denote by ι_α and ι_β the canonical inclusions of A into the multiplier algebras of $A \rtimes_\alpha G$ and $A \rtimes_\beta G$, respectively. Then*

(a) *The actions α and β are exterior equivalent if and only if there is an isomorphism Φ: $A \rtimes_\alpha G \to A \rtimes_\beta G$, intertwining the dual actions $\hat{\alpha}$ and $\hat{\beta}$ of $\hat{G}$, such that (after extending Φ to the multiplier algebra) the diagram*

$$\begin{array}{ccc} & \overset{\iota_\alpha}{\nearrow} & M(A \rtimes_\alpha G) \\ A & & \downarrow \Phi \\ & \underset{\iota_\beta}{\searrow} & M(A \rtimes_\beta G) \end{array}$$

commutes.

(b) *The actions α and β are exterior equivalent modulo an automorphism (i.e., α is exterior equivalent to $g \mapsto \gamma\beta_g\gamma^{-1}$ for some $\gamma \in \mathrm{Aut}(A)$) if and only if there is an isomorphism Φ: $A \rtimes_\alpha G \to A \rtimes_\beta G$, intertwining the dual actions $\hat{\alpha}$ and $\hat{\beta}$ of $\hat{G}$, such that for some $\gamma \in \mathrm{Aut}(A)$, the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\iota_\alpha} & M(A \rtimes_\alpha G) \\ \gamma \uparrow & & \downarrow \Phi \\ A & \xrightarrow{\iota_\beta} & M(A \rtimes_\beta G) \end{array}$$

commutes. (If γ is inner, this notion of equivalence is no different from exterior equivalence in the strict sense.) □

A more detailed study of the above notions of equivalence will be made in Remark 1.7 below. See also [**32**]. Meanwhile, the following proposition (implicit in [**14**]) explains the connection between the Moore cohomology theory and the notion of exterior equivalence.

THEOREM 0.11. *Let α, β: $G \to \mathrm{Aut}(A)$ be two actions of a second-countable locally compact group G on a separable C^*-algebra A, and assume that $\alpha_g\beta_g^{-1} \in \mathrm{Inn}(A)$ for all $g \in G$. Then α and β are exterior equivalent if and only if a certain naturally defined obstruction class in $H^2(G, C(X, \mathbb{T}))$ vanishes. Here X is the largest Hausdorff quotient of* Prim(A) *and $C(X, \mathbb{T})$ is given the topology of uniform convergence on compacta. The action of G on $C(X, \mathbb{T})$ arises from the action of G on* Prim(A) *induced by either α or β.*

PROOF. The map ν: $g \mapsto \alpha_g\beta_g^{-1}$ satisfies the cocycle identity $\nu(gh) = \nu(g)\beta_g\nu(h)\beta_g^{-1}$, and so by Corollary 0.2 defines a continuous 1-cocycle from G to $U(M(A))/C(\mathrm{Prim}\ A, \mathbb{T})$. By definition, α and β are exterior equivalent if and only if ν can be lifted to a continuous 1-cocycle with values in $U(M(A))$. But the short exact sequence

$$1 \to C(X, \mathbb{T}) \to U(M(A)) \to U(M(A))/C(X, \mathbb{T}) \to 1$$

gives rise to an exact sequence

$$H^1(G, U(M(A))) \to H^1(G, U(M(A))/C(X, \mathbb{T})) \to H^2(G, C(X, \mathbb{T})),$$

which shows that the desired lifting exists if and only if the coboundary of the class of ν vanishes in $H^2(G, C(X, \mathbb{T}))$.

Note. Since $U(M(A))$ is noncommutative in general, $H^1(G, U(M(A)))$ and $H^1(G, U(M(A))/C(X, \mathbb{T}))$ are only pointed sets, not groups. Nevertheless, since $C(X, \mathbb{T})$ is central in $U(M(A))$, there is a (not very) long exact cohomology sequence extending out as far as needed, with the usual properties. See, for instance, [**20**, p. 21 and **21**, pp. 40–41]. □

COROLLARY 0.12. *With notation as in the theorem, if α: $G \to \mathrm{Aut}(A)$ is a group action such that $\alpha_g \in \mathrm{Inn}(A)$ for all $g \in G$, then α is unitary if and only if a naturally defined obstruction in $H^2(G, C(X, \mathbb{T}))$ (where G acts trivially on $C(X, \mathbb{T})$) vanishes.*

PROOF. Take $\beta = \mathrm{id}$. □

COROLLARY 0.13. *Suppose A is a separable continuous-trace algebra with spectrum X such that $H^2(X, \mathbb{Z})$ is countable. (By Lemma 0.3, X compact suffices.) Assume α and β are two actions of a connected second-countable locally compact group G on A that induce the same action on X. Then the only obstruction to exterior equivalence of α and β lies in $H^2(G, C(X, \mathbb{T}))$.*

PROOF. By assumption, $\nu(g) = \alpha_g \beta_g^{-1}$ takes values in $\mathrm{Aut}_{C_0(X)} A$, and ν is continuous since $\mathrm{Aut}_{C_0(X)} A$ is closed in $\mathrm{Aut}(A)$. Since G is connected, the range of ν must lie in the connected component of the identity in $\mathrm{Aut}_{C_0(X)} A$. But by Theorem 0.8, $\mathrm{Inn}(A)$ is open in $\mathrm{Aut}_{C_0(X)} A$, so ν must take values in $\mathrm{Inn}(A)$. Now one can apply Theorem 0.11. □

COROLLARY 0.14. *Suppose A is a separable continuous-trace algebra with compact spectrum X. Then any action α of $\mathbb{R}$ or of a connected, simply connected (semisimple) compact Lie group on A which induces the identity on X is necessarily unitary.*

PROOF. Taking $\beta = \mathrm{id}$ in Corollary 0.13, we see the only obstruction to innerness of α lies in $H^2(G, C(X, \mathbb{T}))$, where G acts trivially on X. If $G = \mathbb{R}$ or G is a compact, simply connected Lie group, then $H^2(G, C(X, \mathbb{T})) = 0$ by [**14**, Theorem 2.6] or by [**21**, Proposition 4]. □

REMARK 0.15. Corollary 0.14 can be extended to the case where X is noncompact, by the following argument. For each $x \in X$ and compact neighborhood K of x, α is unitary over K by 0.14, and thus α is locally inner. Where two compact neighborhoods K_1 and K_2 intersect, implementing representations $G \to U(M(A|_{K_1}))$ and $G \to U(M(A|_{K_2}))$ must differ by a continuous function $K_1 \cap K_2 \to \mathrm{Hom}(G, \mathbb{T})$. When G is semisimple, $\mathrm{Hom}(G, \mathbb{T})$ is trivial so we can certainly patch to get a global implementing representation. In the case $G = \mathbb{R}$, $\mathrm{Hom}(G, \mathbb{T}) \cong \underset{\sim}{\mathbb{R}}$ and so the obstruction to global patching lies in $H^1(X, \underset{\sim}{\mathbb{R}})$. Since $\underset{\sim}{\mathbb{R}}$ is a fine sheaf, the obstruction vanishes and α is unitary. For a more detailed discussion and generalizations, see [**32**]. □

Pull-backs and locally unitary group actions. Finally, we review a number of ideas and results from [**27** and **28**]. The following theorem essentially reduces to Theorem 0.5 in the special case $G = \mathbb{Z}$.

THEOREM 0.16 [**27**, THEOREM 2.2, PROPOSITION 2.5, AND THEOREM 3.8]. (a) *Let A be a C^*-algebra with Hausdorff spectrum, and G a locally compact abelian group. Then for any locally unitary action α: $G \to \mathrm{Aut}(A)$, $(A \rtimes_\alpha G)^\wedge \to \hat{A}$ is a locally trivial principal $\hat{G}$-bundle relative to the dual action of $\hat{G}$. Two locally unitary actions are exterior equivalent if and only if the corresponding bundles are isomorphic.*

(b) *If, further, A is stable and $\hat{A}$ is paracompact, then every locally trivial $\hat{G}$-bundle over $\hat{A}$ arises from a locally unitary action of G on A.* □

Finally, we need the notion of pull-backs of C^*-algebras and of group actions, as introduced in [**28**]. If X and Y are locally compact (Hausdorff) spaces, f: $X \to Y$ is a continuous map, and A is a C^*-algebra with $\mathrm{Prim}(A) = Y$, the *pull-back of A along f* is $f^*A = C_0(X) \otimes_{C_b(Y)} A$. (As usual, C_b denotes bounded continuous functions and C_0 denotes continuous functions vanishing at infinity.) In case

$A = \Gamma_0(E)$, the algebra of sections vanishing at infinity of a continuous field E of C^*-algebras over Y, then $f^*A = \Gamma_0(f^*E)$. When $f\colon X \to Y$ is the bundle projection for some principal G-bundle and $\alpha\colon G \to \mathrm{Aut}_{C_0(Y)} A$ is any G-action on A fixing Y pointwise, one can also define $f^*\alpha\colon G \to \mathrm{Aut}\, f^*A$ to be the diagonal action coming from α on A and the translation action of G on $C_0(X)$. Then one of the main results of [**28**] describes the crossed products that arise in the situation of Theorem 0.16.

THEOREM 0.17 [**28**, PROPOSITION 1.5]. *If A is a C^*-algebra with paracompact spectrum Y, G is a locally compact abelian group, and $\alpha\colon G \to \mathrm{Aut}(A)$ is a locally unitary action, then $(A \rtimes_\alpha G, \hat{\alpha})$ is isomorphic (as a C^*-algebra with action of $\hat{G}$) to $(p^*A, p^*\mathrm{id})$, where $p\colon (A \rtimes_\alpha G)^\wedge \to Y$ is the natural map of Theorem* 0.16. □

1. Automorphism groups which act freely on the spectrum. Let A be a C^*-algebra with Hausdorff spectrum Ω and let G be a locally compact group which acts freely on Ω. We suppose that compact subsets of Ω are *wandering* (as in [**10**, p. 88]), or equivalently, that Ω is a principal G-bundle in the sense of Cartan (see [**23**, Theorem 1.2.9]). A common alternative phrase is to say G acts *properly* on Ω, that is, that the map $(g, x) \mapsto (x, gx)$ is proper from $G \times \Omega$ to $\Omega \times \Omega$. We shall study the crossed product $A \rtimes_\alpha G$ of A by an action α of G which induces the given action of G on the spectrum Ω. Our first result shows that such an α can only exist if the algebra is essentially a pull-back in the sense of [**28**].

THEOREM 1.1. *Let α be an action of a locally compact group G on a C^*-algebra A with Hausdorff spectrum Ω. Suppose that the action of G on Ω induced by α is free and proper, and let $p\colon \Omega \to T = \Omega/G$ be the orbit map. Then*

(1) *inducing representations from A to $A \rtimes_\alpha G$ gives a homeomorphism of T onto $(A \rtimes_\alpha G)^\wedge$, such that the resulting action of $C_b(T)$ on $A \rtimes_\alpha G$ agrees with that inherited from the action of $C_b(T) \subset C_b(\Omega)$ on A;*

(2) *there is an isomorphism of $A \otimes \mathcal{K}(L^2(G))$ onto the pull-back $p^*(A \rtimes_\alpha G)$ which respects the natural actions of $C_b(\Omega)$ and intertwines $\alpha \otimes \mathrm{Ad}\,\rho$ with $p^*\mathrm{id}$ (here λ and ρ are the left and right regular representations of G); and*

(3) *if A has continuous trace, so does $A \rtimes_\alpha G$.*

PROOF. (1) As we mentioned above, the properness of the action of G on Ω implies that T is locally compact and Hausdorff and that each G-orbit in Ω (with its relative topology) is homeomorphic to G. Thus the action is regular in the sense of Green [**11**, §5], and induction gives a homeomorphism of T onto $\mathrm{Prim}(A \rtimes_\alpha G)$ by [**11**, Theorem 24 and the subsequent remark]. Every irreducible representation of $A \rtimes_\alpha G$ factors through $(A/I) \rtimes G$, for I the ideal associated to some orbit in Ω, and an application of [**11**, Theorem 17] to this algebra shows that the crossed product is type I. Thus T is homeomorphic to $(A \rtimes_\alpha G)^\wedge$. If $\pi \in \hat{A}$ and $f \in C_b(T)$, a simple calculation shows that for $b \in C_c(G, A)$,

$$\mathrm{Ind}\,\pi(fb) = f(p(\pi))\mathrm{Ind}\,\pi(b).$$

This immediately gives the last part of (1).

(2) There is an isomorphism Ψ of $A \otimes \mathcal{K}(L^2(G))$ onto $E = (C_0(G) \otimes A) \rtimes_{\lambda \otimes \alpha} G$ which carries the action $\alpha \otimes \operatorname{Ad} \rho$ into $\rho \otimes \mathrm{id}$; further, the proof of this fact in [**12**, §2] shows that the isomorphism carries the representation $\pi \otimes \mathrm{id}$ of $A \otimes \mathcal{K}$ into the representation of the imprimitivity algebra E induced from π. We shall show there is an equivariant isomorphism

$$(E, \rho \otimes \mathrm{id}) \cong \left(C_0(\Omega) \otimes_{C_b(T)} (A \rtimes_\alpha G), \tau \otimes_{C_b(T)} \mathrm{id}\right) =_{\mathrm{def}} \left(p^*(A \rtimes_\alpha G), p^*\mathrm{id}\right),$$

where τ is the canonical action of G on $C_0(\Omega)$ and $\rho \otimes \mathrm{id}$ acts through its natural action on $C_0(G) \otimes A$, which commutes with $\lambda \otimes \alpha$ and so passes to an action on E.

We can view A as the algebra $\Gamma_0(F)$ of sections vanishing at infinity of a continuous field F of elementary C^*-algebras over Ω [**4**, 10.5.2]. Let

$$\Delta = \Omega \times_T \Omega =_{\mathrm{def}} \{(x, y) \in \Omega \times \Omega;\ p(x) = p(y)\}.$$

Then if q_1: $\Delta \to \Omega$ and q_2: $G \times \Omega \to \Omega$ in each case denotes projection on the second factor, we may identify $C_0(\Omega) \otimes_{C_b(T)} A$ with $\Gamma_0(q_1^* F)$ and $C_0(G, A)$ with $\Gamma_0(q_2^* F)$. Since ρ: $\Omega \to T$ is a Cartan principal bundle, the map ν: $\Delta \to G \times \Omega$ defined by $\nu(g \cdot y, y) = (g^{-1}, y)$ is a homeomorphism. We define an isomorphism Φ of $C_0(G, A)$ onto $C_0(\Omega) \otimes_{C_b(T)} A$ by $\Phi(f) = f \circ \nu$, $f \in \Gamma_0(q_2^* F)$; this makes sense since $q_2 \circ \nu = q_1$. Each automorphism α_s induces isomorphisms $\alpha_{s,y}$: $F_{s^{-1}y} \to F_y$, and a calculation shows

$$\Phi(\lambda_s \otimes \alpha_s(f))(g \cdot y, y) = \alpha_{s,y}\left(f(s^{-1}g^{-1}, s^{-1}y)\right) = (\mathrm{id} \otimes \alpha_s)(\Phi(f))(gy, y).$$

Thus Φ extends to an isomorphism

$$\Phi_1\colon C_0(G, A) \rtimes_{\lambda \otimes \alpha} G \to \left(C_0(\Omega) \otimes_{C_b(T)} A\right) \rtimes_{\mathrm{id} \otimes \alpha} G \cong p^*(A \rtimes_\alpha G).$$

It is easily checked that Φ_1 is our desired isomorphism $E \to p^*(A \rtimes_\alpha G)$, so $\Phi_1 \circ \Psi$ is an isomorphism of $A \otimes \mathcal{K}$ with $p^*(A \rtimes_\alpha G)$ which sends $\alpha \otimes \operatorname{Ad} \rho$ into $p^*\mathrm{id}$. Since $\nu^{-1}(e, \pi) = (\pi, \pi)$, the induced map on spectra is

$$\pi \mapsto \operatorname{Ind}_A^E \pi \mapsto \pi \otimes \operatorname{Ind}_A^{A \rtimes G} \pi,$$

which via the identification of T with $(A \rtimes G)^\wedge$ in (1) is the natural homeomorphism of Ω onto $p^*(A \rtimes_\alpha G)^\wedge$. This establishes (2).

(3) is an immediate consequence of (2) together with a special case of the following lemma. □

LEMMA 1.2. *Let Ω and T be locally compact Hausdorff spaces and p: $\Omega \to T$ a continuous and open surjection. If B is a C^*-algebra with spectrum T and p^*B has continuous trace, then so does B.*

PROOF. We may suppose $B = \Gamma_0(E)$, where E is a continuous field of (elementary) C^*-algebras over T. Then $p^*B = \Gamma_0(p^*E)$. Choose $t \in T$, $x \in p^{-1}(t)$, and U some compact neighborhood of x in Ω. Choose a continuous self-adjoint section b of E which is a rank-one projection at t. Then for t' close to t, $\|b(t')^2 - b(t')\|$ will be small, hence, modifying b by spectral calculus if necessary, we may assume $b(t')$ is a selfadjoint projection for all $t' \in V$, V some neighborhood of t. Since p was assumed open, without loss of generality we may assume $p(U) \supseteq V$. Now if E does not have a local section of rank-one projections at t, there must be a net $t_i \to t$ in V with $\operatorname{rk} b(t_i) \geqslant 2$ for all i. Choose $x_i \in U$ with $t_i = p(x_i)$. Since U was compact, passing to a subnet we may assume x_i converges to some $x_0 \in U$, and by continuity of p, $p(x_0) = t$. Now $b \circ p$ is a continuous section of p^*E over U, and if p^*B has continuous trace, we must have

$$\operatorname{tr} b \circ p(x_i) = \operatorname{tr} b(t_i) \to \operatorname{tr} b \circ p(x_0) = \operatorname{tr} b(t) = 1.$$

(Although it may not be immediately obvious that $b \circ p$ is a "continuous-trace element" near x_0, this follows by approximating by another section of p^*E, agreeing with $b \circ p$ at x_0 and equal to a rank-one projection everywhere in a neighborhood.) Since we assumed $\operatorname{rk} b(t_i) \geqslant 2$ for all i, this is a contradiction. Thus E must have a local section of rank-one projections around every point of T. By [**4**, Proposition 4.5.4] or [**24**, Theorem 6.1.11], B has continuous trace. □

We have completed the proof of Theorem 1.1. In the case of stable algebras, the following corollary is slightly more convenient.

COROLLARY 1.3. *If, in addition to the hypotheses of Theorem* 1.1, A *is stable, there is an isomorphism of* A *onto* $p^*(A \rtimes_\alpha G)$ *which takes* α *into an action exterior equivalent to* $p^*\mathrm{id}$.

PROOF. This follows immediately from the theorem and the next lemma, since $\alpha \otimes \operatorname{Ad} \rho$ and $\alpha \otimes \mathrm{id}$ are exterior equivalent. □

LEMMA 1.4. *If* $\alpha: G \to \operatorname{Aut} A$ *is a group action on a stable* C^**-algebra* A, *there is an isomorphism* $\nu: A \to A \otimes \mathcal{K}$ *such that* $\nu^{-1} \circ (\alpha \otimes \mathrm{id}) \circ \nu$ *is exterior equivalent to* α.

PROOF. Choose a rank-one projection $e \in \mathcal{K}$ and an isomorphism θ of $A \otimes M_2(\mathbf{C})$ with $A \otimes \mathcal{K}$ such that $\theta(a \otimes e_{11}) = a \otimes e$ for $a \in A$. Then define ν by

$$A \xrightarrow{\cong} A \otimes e_{22} \xrightarrow{\theta} A \otimes (1-e)\mathcal{K}(1-e) \cong A \otimes \mathcal{K}((1-e)(\mathcal{H})).$$

Note that $\alpha \otimes \mathrm{id}_{\mathcal{K}(\mathcal{H})}$ leaves $A \otimes \mathcal{K}((1-e)\mathcal{H})$ invariant, and

$$\theta^{-1} \circ \left(\alpha \otimes \mathrm{id}_{\mathcal{K}(\mathcal{H})}\right) \circ \theta$$

is an automorphism of $A \otimes M_2$ which is equal to α on the upper left-hand corner and to $\nu^{-1} \circ (\alpha \otimes \mathrm{id}_{K((1-e)\mathcal{H})}) \circ \nu$ on the bottom right. The result follows from [**24**, 8.11.2]. □

REMARK. The hypothesis that G act properly on Ω is automatic if G is compact or if the bundle $p: \Omega \to \Omega/G$ is locally trivial. Conversely, if G is a Lie group, any free proper action yields a locally trivial bundle by [**23**, Theorem 4.1]. (The case of a compact Lie group is much easier and may be found in [**6**].)

Suppose that A is a separable continuous-trace C^*-algebra with $\hat{A} = \Omega$ and α: $G \to \operatorname{Aut} A$ is an action such that the orbit map $p: \Omega \to T$ is a principal G-bundle. In light of [**28**, Proposition 1.4], our theorem asserts that $\delta(A) = \delta(A \otimes \mathcal{K})$ belongs to the range of p^*: $H^3(T, \mathbf{Z}) \to H^3(\Omega, \mathbf{Z})$, and then $A \otimes \mathcal{K} \cong p^*B$ whenever B is a stable continuous-trace algebra with $p^*\delta(B) = \delta(A)$. Now we know that in at least one such realization $\alpha \otimes \mathrm{id}$ is equivalent to $p^*\mathrm{id}$, but this only works if we have made the correct choice $A \rtimes_\alpha G$ for B, and this may well not be the obvious one. For example, if $A = p^*C$ and $\alpha = p^*\gamma$ for some $\gamma\colon G \to \mathrm{Aut}_{C_0(T)} C$, examples are given in [**28**, Corollary 3.6] with $p^*C \rtimes_{p^*\gamma} G$ not Morita equivalent to C. In such cases, the necessary B is not the algebra C we started with. So Theorem 1.1 suggests two problems:

(a) First of all, given $A = p^*B$ and $\alpha\colon G \to \mathrm{Aut}\, A$ inducing the bundle p on spectra, when is α equivalent to $p^*\gamma$ for some reasonable automorphism group γ of B?

(b) Secondly, can we describe the isomorphism class of a crossed product $p^*B \rtimes_{p^*\gamma} G$ in terms of topological data associated with p, B, and γ?

Our next result is an answer to the second question in the case where γ is locally unitary, which by [**32**, Theorem 2.5] is less of a restriction than one might think. We shall later show how (b) helps with (a)—for further discussion, see also [**32**, §4].

We intend, then to give a formula for the Dixmier-Douady class of $p^*B \rtimes_{p^*\gamma} G$ for G abelian and γ locally unitary. To state it, we need to set up some notation and a pairing $\langle\,\cdot\,,\cdot\rangle\colon\ H^1(T,\hat{\underset{\sim}{G}}) \times H^1(T,\underset{\sim}{G}) \to H^3(T,\mathbb{Z})$. If G is an abelian locally compact group and $\gamma\colon G \to \mathrm{Aut}\, B$ is locally unitary, let $\zeta(\gamma)$ be the class of the corresponding bundle $(B \rtimes_\gamma G)^\wedge \to \hat{B}$ (as in [**27**] and our Theorem 0.16) in $H^1(\hat{B},\hat{\underset{\sim}{G}})$. If $p\colon \Omega \to T$ is a principal G-bundle, write $[p]$ for its class in $H^1(T,\underset{\sim}{G})$. To construct the pairing, let $c \in H^1(T,\hat{\underset{\sim}{G}})$ and $d \in H^1(T,\underset{\sim}{G})$. Then the cup product $c \cup d$ (defined in [**8**, §II.6.6]) lies in $H^2(T,\hat{\underset{\sim}{G}} \otimes \underset{\sim}{G})$ which maps to $H^2(T,\underset{\sim}{\mathbb{T}})$ under the map $\underset{\sim}{G} \otimes \underset{\sim}{G} \to \underset{\sim}{\mathbb{T}}$ induced by the dual pairing $\hat{G} \times G \to \mathbb{T}$. We let $\langle c, d\rangle$ be the image of $c \cup d$ in $H^2(T,\underset{\sim}{\mathbb{T}})$, transported to $H^3(T,\mathbb{Z})$ via the usual isomorphism $H^2(T,\underset{\sim}{\mathbb{T}}) \to H^3(T,\mathbb{Z})$. In terms of Čech cocycles, if c and d are represented by cocycles $\gamma_{ij}\colon\ N_{ij} \to \hat{G}$, $g_{ij}\colon\ N_{ij} \to G$ relative to some cover $\{N_j\}$ of T, $\langle c, d\rangle$ is represented by the cocycle in $Z^2(\{N_j\},\underset{\sim}{\mathbb{T}})$

$$h_{ijk}\colon N_{ijk} \to \mathbb{T} \quad \text{where } h_{ijk}(t) = \langle \gamma_{ij}(t), g_{jk}(t)\rangle.$$

THEOREM 1.5. *Let B be a continuous-trace algebra with paracompact spectrum T, let $\gamma\colon G \to \mathrm{Aut}\, B$ be a locally unitary abelian automorphism group, and let $p\colon \Omega \to T$ be a locally trivial principal G-bundle. Then the crossed product $p^*B \rtimes_{p^*\gamma} G$ has continuous trace and*

$$\delta\big(p^*B \rtimes_{p^*\gamma} G\big) = \delta(B) + \langle \zeta(\gamma), [p]\rangle.$$

PROOF. We have already seen that $p^*B \rtimes G$ has continuous trace (Theorem 1.1(3)). In [**28**, Theorem 2.5], it was shown that the crossed product is strongly Morita equivalent to an algebra there denoted $GC(\Omega, B)^\alpha/I$. Here $\alpha = \tau \otimes \gamma$ where τ is the translation action of G on $C_0(\Omega)$, and $GC(\Omega, B)$ is a certain algebra of bounded continuous functions $\Omega \to B$. It will be enough to calculate $\delta(GC(\Omega, B)^\alpha/I)$. To this end, we fix a covering of T by open sets M_i, whose closures N_i are compact, with the following properties:

(1) There are continuous fields of Hilbert spaces H_i over N_i, isomorphisms θ_i of $B|_{N_i}$ onto $\Gamma(\mathfrak{A}(H_i))$, and isomorphisms v_{ij} of $H_j|_{N_{ij}}$ onto $H_i|_{N_{ij}}$ such that

$$\mathrm{Ad}\big(v_{ij}(t)\big) = \theta_i(t) \circ \theta_j^{-1}(t) \quad \text{for } t \in N_{ij}.$$

Then there is a Čech cocycle $\mu \in Z^2((N_i), \mathbb{T})$ with

$$v_{ij}(t)v_{jk}(t) = \mu_{ijk}(t)v_{ik}(t) \quad \text{for } t \in N_{ijk},$$

and $\delta(B) \in H^2(T, \mathbb{T})$ is by definition the class of this cocycle [4, 10.7.11–10.7.14].

(2) There are G-bundle isomorphisms h_i: $p^{-1}(N_i) \to N_i \times G$ and a cocycle $\lambda \in Z^1((N_i), \underset{\sim}{G})$ with $h_i \circ h_j^{-1}(t, g) = (t, g\lambda_{ij}(t))$, and $[p] \in H^1(T, \underset{\sim}{G})$ is the class of this cocycle.

(3) There are strictly continuous homomorphisms u^i of G into $M(\Gamma(\mathfrak{A}(H_i)))$ such that $\mathrm{Ad}(u^i_g) = \theta_i \circ \gamma_g \circ \theta_i^{-1}$. It follows that $\mathrm{Ad}(v_{ij}u^j_g) = \mathrm{Ad}(u^i_g v_{ij})$ on N_{ij}, so that there are also continuous maps χ_{ij}: $N_{ij} \to \hat{G}$ satisfying

$$v_{ij}(t)u^j_g(t) = \chi_{ij}(t)(g)u^i_g(t)v_{ij}(t) \quad \text{for } t \in N_{ij}.$$

(Recall $M(\Gamma(\mathfrak{A}(H_i)))$ consists of fields over N_i of bounded operators on H_i.) The χ_{ij} form a cocycle which realizes $\zeta(\gamma) \in H^1(T, \underset{\sim}{\hat{G}})$. (See [27, pp. 223–224]; the u^i there are slightly different.)

As in the proof of [28, Proposition 3.3], for each trivialization h_i of p the map Ψ_i: $GC(\Omega, B)^\alpha \to B|_{N_i}$ defined by

$$\Psi_i(b)(t) = b\big(h_i^{-1}(t, e)\big)(t)$$

induces an isomorphism of $[GC(\Omega, B)^\alpha/I]|_{N_i}$ onto $B|_{N_i}$. It is routine to verify that $\Psi_j(t) = \gamma_{\lambda_{ij}(t)} \circ \Psi_i(t)$. We now define $\Phi_i = \theta_i \circ \Psi_i$, so that Φ_i trivializes $GC(\Omega, B)^\alpha/I$ over N_i. More calculations give

$$\Phi_i(t) \circ \Phi_j(t)^{-1} = \mathrm{Ad}\Big(u^i_{\lambda_{ij}(t)}(t)v_{ij}(t)\Big),$$

so that $\delta(GC(\Omega, B)^\alpha/I)$ is represented by the cocycle ν with

$$u^i_{\lambda_{ij}(t)}(t)v_{ij}(t)u^j_{\lambda_{jk}(t)}(t)v_{jk}(t) = \nu_{ijk}(t)u^i_{\lambda_{ik}(t)}(t)v_{ij}(t), \qquad t \in N_{ijk}.$$

A gory calculation shows that

$$\nu_{ijk}(t) = \langle \chi_{ij}(t), \lambda_{jk}(t)\rangle \mu_{ijk}(t),$$

and the result follows. □

COROLLARY 1.6. *Suppose that B_1, B_2 are stable separable continuous-trace C^*-algebras with the same spectrum T, that G is a second-countable locally compact abelian group, that γ_i: $G \to \mathrm{Aut}\, B_i$ $(i = 1, 2)$ are locally unitary, and that ρ: $\Omega \to T$ is a locally trivial principal G-bundle. Then $p^*\gamma_1$ is exterior equivalent to $\nu^{-1} \circ p^*\gamma_2 \circ \nu$, for some $C_b(\Omega)$-isomorphism ν of p^*B_1 onto p^*B_2, if and only if*

$$\delta(B_1) - \delta(B_2) = \langle \zeta(\gamma_1) - \zeta(\gamma_2), [p]\rangle.$$

PROOF. First suppose that the condition holds. Then

$$\delta(p^*B_1) - \delta(p^*B_2) = p^*(\delta(B_1) - \delta(B_2)) \in \langle p^*H^2(T, \mathbb{Z}), p^*[p]\rangle,$$

but $p^*[p] = 0$ (since p^*p is a trivial bundle over Ω), so $\delta(p^*B_1) = \delta(p^*B_2)$ and p^*B_1 and p^*B_2 are $C_b(\Omega)$-isomorphic. Furthermore, the theorem and the condition give

$$\delta\big(p^*B_1 \rtimes_{p^*\gamma_1} G\big) = \delta\big(p^*B_2 \rtimes_{p^*\gamma_2} G\big),$$

and since both crossed products are stable (say by Lemma 1.4) and have spectrum canonically isomorphic to T, we also have a $C_b(T)$-isomorphism of $p^*B_1 \rtimes_{p^*\gamma_1} G$ onto $p^*B_2 \rtimes_{p^*\gamma_2} G$. Thus there is a $C_b(\Omega)$-isomorphism between their pull-backs, sending $p^*\,\mathrm{id}$ to $p^*\,\mathrm{id}$, and the exterior equivalence of $p^*\gamma_1$ with a conjugate of $p^*\gamma_2$ follows from Corollary 1.3 (applied twice).

The necessity of the condition is an immediate consequence of Theorems 0.10 and 1.5. □

REMARK 1.7. This corollary gives a topological criterion for distinguishing automorphism groups of the form $p^*\gamma$ modulo the equivalence relation generated by exterior equivalence and conjugacy under *spectrum-fixing* automorphisms. (Recall the discussion accompanying Theorem 0.10.) This equivalence relation is strictly weaker than exterior equivalence; in fact, it is shown in [**32**, Proposition 4.2] that if $B_1 = B_2$ in the situation of Corollary 1.6, then $p^*\gamma_1$ is exterior equivalent to $p^*\gamma_2$ if and only if $\zeta(\gamma_1) = \zeta(\gamma_2)$, whereas we have shown that $p^*\gamma_1$ is exterior equivalent to a suitable $\nu \circ p^*\gamma_2 \circ \nu^{-1}$ if and only if $\langle \zeta(\gamma_1) - \zeta(\gamma_2), [p]\rangle = 0$.

To understand this distinction, note that if α is an action of G on a C^*-algebra A, and if $\nu \in \mathrm{Aut}_{C_0(\hat{A})} A$, then α and $\nu \circ \alpha \circ \nu^{-1}$ will be exterior equivalent exactly when the cocycle from G to $\mathrm{Aut}_{C_0(\hat{A})} A$ (for the action given by $g \cdot \gamma = \alpha_g \gamma \alpha_g^{-1}$):

$$c\colon g \mapsto \nu\alpha_g\nu^{-1}\alpha_g^{-1}$$

is implemented by a continuous map $u\colon G \to U(M(A))$ which is a cocycle for the action given by $g \cdot v = \alpha_g(v)$. There are two possible obstructions to the existence of such a cocycle u:

(a) c need not take values in $\mathrm{Inn}\, A$, and

(b) c can map into $\mathrm{Inn}\, A$ without lifting to $U(M(A))$.

We shall show by example that both (a) and (b) are genuine obstructions, even when G is an abelian Lie group and A is a continuous-trace algebra.

For our first example, we take $G = \mathbb{Z}$, so we might as well be looking at single automorphisms α. The problem is then whether the commutator $\nu\alpha\nu^{-1}\alpha^{-1}$ is inner. Let $A = C(S^2, \mathcal{K})$, let ν be a $C(S^2)$-automorphism of A which is not inner, and let α be the automorphism induced by the antipodal homeomorphism of S^2 with itself. By Theorem 0.5 and Remark 0.6,

$$\mathrm{Out}(A) \cong H^2(S^2, \mathbb{Z}) \rtimes \mathrm{Homeo}(S^2).$$

Since the antipodal map acts by -1 on $H^2(S^2)$, conjugation by α in $\mathrm{Out}(A)$ sends $\zeta(\nu^{-1}) = -\zeta(\nu)$ to $\zeta(\nu)$, and so $\zeta(\nu\alpha\nu^{-1}\alpha^{-1}) = \zeta(\nu^2) \neq 0$ and $\nu\alpha\nu^{-1}\alpha^{-1}$ is not inner by Theorem 0.5. Thus (a) happens in this case.

For our second example, we take $G = \mathbb{T}^2$, $A = C(\mathbb{T}^2, \mathcal{K}(\mathcal{H}))$, α to be the translation action of G on A (i.e., $\alpha_g(a)(z) = a(g^{-1}z)$). An outer $C(\mathbb{T}^2)$-automorphism ν of A is given by a continuous map $\psi\colon \mathbb{T}^2 \to PU(\mathcal{H})$ which does not lift to $U(\mathcal{H})$. (Note that since $PU(\mathcal{H})$ is a $K(\mathbb{Z}, 2)$-space if $\mathcal{H}$ is infinite-dimensional, $[\mathbb{T}^2, PU(\mathcal{H})] \cong H^2(\mathbb{T}^2, \mathbb{Z})$, and ψ lifts if and only if it represents the trivial element of H^2.) Then $\nu\alpha_g\nu^{-1}\alpha_g^{-1}$ is given by the continuous map $z \mapsto \psi(z)\psi(g^{-1}z)^{-1}$. If $u\colon G \times \mathbb{T}^2 \to U(\mathcal{H})$ were a continuous map with

$$\mathrm{Ad}\, u(g, z) = \psi(z)\psi(g^{-1}z)^{-1} \quad \text{for } g \in G \text{ and } z \in \mathbb{T}^2,$$

then in particular $g \mapsto u(g,1)$ would be a continuous lifting of $g \mapsto \psi(1)\psi(g^{-1})^{-1}$, which cannot exist since ψ does not lift. So such a u cannot exist and obstruction (b) occurs here. Note that in this example, if $p\colon \mathbb{T}^2 \to \mathrm{pt}$, then $A \cong p^*\mathscr{K}$ and $\alpha = p^*\,\mathrm{id}$, so this is the situation that arises in Corollary 1.6. □

Let B and $p\colon \Omega \to T$ be as in Corollary 1.6. We turn now to the question of whether a given automorphism group $\alpha\colon G \to \operatorname{Aut} p^*B$ which induces the bundle p on $\Omega = (p^*B)^\wedge$ is equivalent to one of the form $p^*\gamma$ for some locally unitary group $\gamma\colon G \to \operatorname{Aut} B$. (Here "equivalent" is in the sense just discussed; if equivalence were to mean exterior equivalence, the answer is different and is discussed in [**32**].) Corollary 1.6 and Theorem 1.1 show that this is equivalent to asking whether

$$\delta(B) - \delta(p^*B \rtimes_\alpha G) = \langle \zeta(\gamma), [p]\rangle$$

for some such γ. Now by Theorem 0.5(b), $\zeta(\gamma)$ can be any class in $H^1(T, \underset{\sim}{\hat{G}})$, while by Theorem 1.1,

$$p^*\big(\delta(p^*B \rtimes_\alpha G)\big) = \delta\big(p^*(p^*B \rtimes_\alpha G)\big) = \delta(p^*B) = p^*\delta(B).$$

Hence α will always have the required form if the map $c \mapsto \langle c, [p]\rangle$ carries $H^1(T, \underset{\sim}{\hat{G}})$ *onto* the kernel of $p^*\colon H^3(T, \mathbb{Z}) \to H^3(\Omega, \mathbb{Z})$. This criterion quickly gives some positive answers to our question (for example, when $G = \mathbb{R}^n$, all principal bundles are trivial and p^* is an isomorphism), and can also be used to show that the answer in general is negative.

PROPOSITION 1.8. *Suppose that B is a separable stable continuous-trace C^*-algebra and that $\mathbb{T}$ acts freely on a locally compact space Ω with $\Omega/\mathbb{T} = T$ homeomorphic to $\hat{B}$. If $\alpha\colon\mathbb{T} \to \operatorname{Aut} p^*B$ is an automorphism group which induces the bundle $p\colon \Omega \to T$ on $\Omega = (p^*B)^\wedge$, then there is a locally unitary automorphism group $\gamma\colon \mathbb{T} \to \operatorname{Aut} B$ and a $v \in \operatorname{Aut}_{C_0(T)} p^*B$ such that α is exterior equivalent to $v \circ p^*\gamma \circ v^{-1}$.*

PROOF. Recall that p is automatically locally trivial by Gleason's Theorem [**6**], so our theory applies. Furthermore, under the usual identification of $H^1(T, \underset{\sim}{\mathbb{T}})$ with $H^2(T, \mathbb{Z})$, $\langle\,\cdot\,,\cdot\rangle$ becomes just the ordinary cup product in Čech cohomology

$$\cup\colon H^1(T,\mathbb{Z}) \times H^2(T,\mathbb{Z}) \to H^3(T,\mathbb{Z}).$$

But the Gysin exact sequence for the bundle $p\colon \Omega \to T$ (see for instance [**36**, Theorem VII.5.12] or [**33**, Theorem VII.7.11]—this works in Čech cohomology just as for singular cohomology on finite complexes, via [**8**, §II.4.17])

$$\cdots \to H^1(T,\mathbb{Z}) \xrightarrow{\cup[p]} H^3(T,\mathbb{Z}) \xrightarrow{p^*} H^3(\Omega,\mathbb{Z}) \xrightarrow{p_!} \cdots$$

says the image of $\langle\,\cdot\,,[p]\rangle$ is exactly the kernel of p^*. □

REMARK. The Gysin sequence shows at the same time that there can be many γ's with $p^*\gamma$ equivalent to α. For the indeterminancy in $v \in \operatorname{Aut}_{C_0(\Omega)} p^*B$, see [**32**, Theorem 4.4].

EXAMPLE 1.9. We shall now construct a stable continuous-trace C^*-algebra A with (compact) spectrum Ω, a principal $\mathbb{T}^2$-bundle $p\colon \Omega \to T$, a stable continuous-trace algebra B with spectrum T such that $p^*B \cong A$ as $C(\Omega)$-algebras, and an automorphism group $\alpha: \mathbb{T}^2 \to \operatorname{Aut} A$ which induces the given $\mathbb{T}^2$-action on Ω but is not equivalent to $p^*\gamma$ for any locally unitary $\gamma: \mathbb{T}^2 \to \operatorname{Aut} B$.

We begin by observing that it is enough to construct $p: \Omega \to T$ and an element $\eta \in H^3(T, \mathbb{Z})$ with $p^*\eta = 0$ which does not have the form $\langle c, [p] \rangle$ for any $c \in H^1(T, \mathbb{Z}^2)$. For given such a cohomology class η, we can take $A = C(\Omega, \mathcal{K}(\mathcal{H}))$ and C to be the separable stable continuous-trace algebra over T with $\delta(C) = \eta$. Then there is a $C(\Omega)$-isomorphism ψ of A onto p^*C (since $p^*\eta = \delta(A)$) and we can define $\alpha = \psi^{-1} \circ p^* \operatorname{id}_C \circ \psi$. The algebra A is also $C(\Omega)$-isomorphic to p^*B, where $B = C(T, \mathcal{K}(\mathcal{H}))$, and if α were equivalent to $p^*\gamma$, γ a locally unitary action on B, we would have

$$p^*B \rtimes_{p^*\gamma} G \cong A \rtimes_\alpha G \cong p^*C \rtimes_{p^* \operatorname{id}_C} G \cong C$$

(by the dual version of Theorem 0.17). By Theorem 1.5, this would imply

$$\eta = \delta(C) = \delta\left(p^*B \rtimes_{p^*\gamma} G\right) = \langle \zeta(\gamma), [p] \rangle,$$

contradicting the hypothesis on η.

To obtain a specific example with the desired properties, let $T = PU(2) \times PU(2) \cong \mathbb{RP}^3 \times \mathbb{RP}^3$, and let $\Omega = U(2) \times U(2)$, with $p: \Omega \to T$ the usual quotient map. Since $\pi_1(PU(2)) \cong \mathbb{Z}_2$ and $\pi_1(U(2)) \cong \mathbb{Z}$, we have

$$H^1(PU(2), \mathbb{Z}) = 0, \quad H^2(PU(2), \mathbb{Z}) = \mathbb{Z}_2, \quad \text{and} \quad H^3(PU(2), \mathbb{Z}) \cong \mathbb{Z},$$

while

$$H^1(U(2), \mathbb{Z}) \cong H^3(U(2), \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad H^2(U(2), \mathbb{Z}) = 0.$$

By the Künneth Theorem,

$$\begin{cases} H^1(T, \mathbb{Z}) = 0, \\ H^3(T, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus \operatorname{Tor}\left(H^2(PU(2), \mathbb{Z}),\ H^2(PU(2), \mathbb{Z})\right) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2, \text{ and} \\ H^3(\Omega, \mathbb{Z}) \cong \mathbb{Z}^2. \end{cases}$$

Let η be the nontrivial torsion element of $H^3(T, \mathbb{Z})$. Since $H^3(\Omega, \mathbb{Z})$ is torsion-free, $p^*\eta = 0$. On the other hand, η cannot be of the form $\langle c, [p] \rangle$ with $c \in H^1(T, \mathbb{Z}^2) \cong H^1(T, \mathbb{Z})^2$, since $H^1(T, \mathbb{Z}) = 0$. □

2. Automorphism groups with a fixed isotropy group for the action on the spectrum. Let $\alpha: G \to \operatorname{Aut} A$ be an action of an abelian group G on a C^*-algebra A with Hausdorff spectrum, and suppose that the isotropy group of every $\pi \in \hat{A}$ is a fixed closed subgroup H of G. The first step in understanding $A \rtimes_\alpha G$ is to describe its primitive ideal space. This can be done fairly generally using Green's version of the Mackey machine [**11, 12**], assuming the action of G is sufficiently well behaved. Specifically, we shall assume that $\hat{A}$ is a principal G/H-bundle over $\hat{A}/G$ and that $\alpha|_H$ is locally unitary. Neither hypothesis is really that restrictive; the first follows from [**23**, Theorem 4.1] when G/H is a Lie group acting properly on $\hat{A}$, and the second follows from [**32**, Theorem 2.5] when G and A are separable, H is compactly generated and all the Mackey obstructions vanish in $H^2(H, \mathbb{T})$. In these cases, the spectrum of $A \rtimes_\alpha G$ is a principal bundle for the dual action of $\hat{H} \cong \hat{G}/H^\perp$, and we have once again the possibility of obtaining information about $A \rtimes_\alpha G$ in terms of topological invariants associated with A, $\alpha|_H$, and the quotient map $\hat{A} \to \hat{A}/G$. For simplicity of notation, we shall write $A \rtimes_\alpha H$ for $A \rtimes_{\alpha|_H} H$.

PROPOSITION 2.1. *Let G be a locally compact group acting on a C^*-algebra in such a way that the isotropy group of each primitive ideal is the same closed (normal) subgroup H. Suppose that (G, A) is regular in the sense of* [**11**, *p.* 223], *and that* Prim$A$ *is Hausdorff. Then restriction of representations defines a continuous surjection* Res: $\mathrm{Prim}(A \rtimes_\alpha H) \to \mathrm{Prim}(A)$ *and induction defines a continuous surjection* Ind: $\mathrm{Prim}(A \rtimes_\alpha H) \to \mathrm{Prim}(A \rtimes_\alpha G)$. *Let p*: $\mathrm{Prim}\,A \to (\mathrm{Prim}\,A)/G$ *be the quotient map and let q be the map sending some $J \in \mathrm{Prim}(A \rtimes_\alpha G)$ to the G-orbit on which the restriction of J to A lives. Then we have a commutative diagram*

$$\begin{array}{ccccc} & & \mathrm{Prim}(A \rtimes_\alpha H) & & \\ & \mathrm{Ind} \swarrow & & \searrow \mathrm{Res} & \\ \mathrm{Prim}(A \rtimes_\alpha G) & & & & \mathrm{Prim}\,A \\ & q \searrow & & \swarrow p & \\ & & (\mathrm{Prim}\,A)/G & & \end{array}$$

and Ind *passes to a homeomorphism of the space Q of G-quasi-orbits in* $\mathrm{Prim}(A \rtimes_\alpha H)$ *onto* $\mathrm{Prim}(A \rtimes_\alpha G)$.

PROOF. We first observe that since H acts trivially on Prim A, the quasi-orbit space for this action is just Prim A itself. Thus in particular (H, A) is quasi-regular [**11**, Corollary 19] and restriction maps $\mathrm{Prim}(A \rtimes_\alpha H)$ into Prim A; it is continuous by [**11**, Proposition 9] and surjective since $\mathrm{Res}(\mathrm{Ind}\, I) = I$ for trivial group actions [**11**, Proposition 11].

To get the properties of Ind we wish to apply [**11**, Theorem 24] on essentially free actions, so we use the isomorphism R: $A \rtimes_\alpha G \to C^*(G, A \rtimes_\alpha H, \mathscr{T}^H)$ of [**11**, Proposition 1] to reduce to that case. By [**11**, Proposition 20] every primitive ideal of $A \rtimes_\alpha G$ is induced from $A \rtimes_\alpha H$, and the isomorphism R respects induction [**11**, Proposition 7], so it follows that every primitive ideal J of $C^*(G, A \rtimes_\alpha H, \mathscr{T}^H)$ is induced from some primitive ideal I of $C^*(H, A \rtimes_\alpha H, \mathscr{T}^H) \cong A \rtimes_\alpha H$. By [**11**, Proposition 11(ii)] we have

$$\mathrm{Res}\, J = \mathrm{Res}(\mathrm{Ind}\, I) = \bigcap_{s \in G} s \cdot I,$$

which says in particular that J lives over a quasi-orbit and $(G, A \rtimes_\alpha H, \mathscr{T}^H)$ is quasi-regular. In the notation of [**11**], the action of G on $A \rtimes_\alpha H$ is given by

$$(s \cdot f)(t) = \Delta_{G,H}(s)\alpha_s(f(s^{-1}ts)),$$

and a simple calculation shows

$$s \cdot (\ker(\pi \times U)) = \ker(s \cdot \pi \times (U \circ \mathrm{Ad}\, s^{-1})),$$

where $\mathrm{Ad}\, s^{-1}(t) = s^{-1}ts$. If $s \in H$, note that

$$\ker(s \cdot \pi \times (U \circ \mathrm{Ad}\, s^{-1})) = \ker(\pi \times U),$$

whereas if $s \notin H$, then $\ker(s \cdot \pi) \neq \ker \pi$. So the isotropy group of each point in $\operatorname{Prim}(A \rtimes_\alpha H)$ is H, or, in other words, $(G, A \rtimes_\alpha H, \mathscr{T}^H)$ is essentially free. We have already checked that $(G, A \rtimes_\alpha H, \mathscr{T}^H)$ is quasi-regular and EH-regular, which is enough to make the proof of [**11**, Theorem 24] go through (see the remark following its proof). We deduce that Ind maps $\operatorname{Prim}(A \rtimes_\alpha H)$ continuously onto $\operatorname{Prim}(A \rtimes_\alpha G)$, and induces a homeomorphism of the quasi-orbit space Q onto $\operatorname{Prim}(A \rtimes_\alpha G)$.

By [**11**, Proposition 9 and Proposition 11], restriction defines a continuous map of $\operatorname{Prim}(A \rtimes_\alpha G)$ into the set ${}^G\mathscr{I}(A)$ of G-invariant ideals of A. The regularity of (G, A) implies that the range of this map is in the image of the canonical map $(\operatorname{Prim} A)/G \to {}^G\mathscr{I}(A)$, and therefore it defines a continuous map of $\operatorname{Prim}(A \rtimes_\alpha G)$ into $(\operatorname{Prim} A)/G$ by the lemma on p. 221 of [**11**]. The map p is continuous by definition of the quotient topology, so it only remains to check that the diagram commutes.

Let $J = \ker(\pi \times U)$ be a primitive ideal of $A \rtimes_\alpha H$. Its image going down the right-hand side of the diagram is the G-orbit of $\ker \pi$ in $\operatorname{Prim} A$. On the other hand

$$\operatorname{Ind} J = \ker\left(\operatorname{Ind}_H^G(\pi \times U)\right),$$

so going down the left-hand side gives the orbit on which $\operatorname{Ind}_H^G(\pi \times U)|_A$ lives. This representation can be realized on a completion of $C_c(G, A) \otimes \mathscr{H}_\pi$, and for $a \in A$,

$$\begin{aligned}\operatorname{Ind}_G^H(\pi \times U)(a) = 0 &\Leftrightarrow \langle (a^*\varphi) \otimes \xi, \psi \otimes \eta \rangle = 0 \quad \forall \xi, \eta \in \mathscr{H}_\pi;\ \varphi, \psi \in C_c(G, A)\\ &\Leftrightarrow \langle (\pi \times U)\left(\langle \psi, a^*\varphi \rangle_{C_c(H,A)}\right)\xi, \eta \rangle = 0 \quad \forall \xi, \eta, \varphi, \psi.\end{aligned}$$

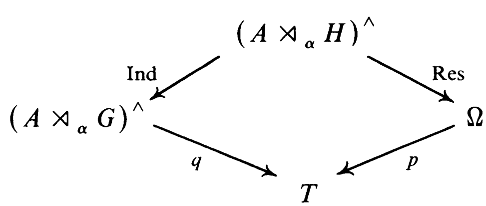
We can rewrite this last inner product as a double integral, which clearly vanishes if $\pi(\alpha_s(a)) = 0$ for all $s \in G$. However, using approximate identities, one can see this condition is also necessary, hence

$$\ker\left(\operatorname{Ind}_H^G(\pi \times U)|_A\right) = G \cdot (\ker \pi),$$

and the diagram commutes as claimed. □

REMARK. The condition that $\operatorname{Prim} A$ is Hausdorff was only used to ensure that (A, H) is quasi-regular, and this is automatic if everything is separable. Although quasi-regularity presumably is automatic even in many cases with A nonseparable and $\operatorname{Prim} A$ only T_0, the Hausdorff case will be more than enough for our purposes.

THEOREM 2.2. *Let A be a C^*-algebra with Hausdorff spectrum Ω, and let α: $G \to \operatorname{Aut} A$ be a locally compact abelian automorphism group such that the isotropy groups for the action of G on Ω are all equal to a fixed subgroup H. Suppose that the quotient map p: $\Omega \to T = \Omega/G$ is a locally trivial principal (G/H)-bundle, and that the restriction of α to H is locally unitary. Then all the maps in the commutative diagram*



of Proposition 2.1 *are principal bundles,* Res *with respect to the dual action of* $\hat{H}$, Ind *for the action of* G/H *inherited from the action of* G *on* $A \rtimes_\alpha H$ *defined by* $s \cdot \nu(t) = \alpha_s(\nu(t))$, *and* q *for the dual action of* $\hat{H} \cong \hat{G}/H^\perp$ *on* $A \rtimes_\alpha G$. *The actions on* $(A \rtimes_\alpha H)^\wedge$ *of* G/H *(with quotient* $(A \rtimes_\alpha G)^\wedge$*) and of* $\hat{H}$ *(with quotient* Ω*) commute. Further,* $A \rtimes_\alpha H$ *is isomorphic to* $\mathrm{Res}^* A$ *and strongly Morita equivalent to* $\mathrm{Ind}^*(A \rtimes_\alpha G)$. *The diagram is involutive with respect to Takai duality, which interchanges the roles of* G *and* $\hat{G}$, G/H *and* $\hat{H}$.

The proof of this will use two lemmas. The first is stated separately merely for future convenience, but the second may be of some independent interest in that it provides many concrete examples of locally unitary actions.

LEMMA 2.3. *If* $\alpha\colon G \to \operatorname{Aut} A$ *is an action of a locally compact abelian group on a* C^**-algebra* A, H *is a closed subgroup of* G, *and* (π, U) *is a covariant representation of* $(A, H, \alpha|_H)$, *then*

$$\mathrm{Ind}_H^G(\pi \times \gamma|_H U) \cong \left(\mathrm{Ind}_H^G(\pi \times U)\right) \circ \hat{\alpha}_\gamma^{-1} \quad \text{for } \gamma \in \hat{G}.$$

PROOF. If L is any representation of $A \rtimes_\alpha H$ on $\mathscr{H}_L$, then $\mathrm{Ind}_H^G L$ acts on the completion V_L of $C_c(G, A) \otimes \mathscr{H}_L$ for the inner product

$$\langle x \otimes \xi, y \otimes \eta \rangle = \left\langle L\left(\langle y, x\rangle_{C_c(H,A)}\right)\xi, \eta\right\rangle.$$

Note that $W\colon x \otimes \xi \mapsto \hat{\alpha}_\gamma(x) \otimes \xi$ defines a unitary operator from $V_{\pi \times U}$ onto $V_{\pi \times \gamma|_H U}$, and for $z \in C_c(G, A)$, we have

$$\begin{aligned}\left[\mathrm{Ind}(\pi \times \gamma|_H U)(z)W\right](x \otimes \xi) &= \left[z^*\hat{\alpha}_\gamma(x)\right] \otimes \xi = \hat{\alpha}_\gamma\left[\hat{\alpha}_\gamma^{-1}(z)^* x\right] \otimes \xi \\ &= W\left[\mathrm{Ind}(\pi \times U)\left(\hat{\alpha}_\gamma^{-1}(z)\right)\right](x \otimes \xi). \quad \square\end{aligned}$$

LEMMA 2.4. *Suppose* A *is a* C^**-algebra with Hausdorff spectrum,* G *is a locally compact abelian group, and* $\alpha\colon G \to \operatorname{Aut} A$ *has* $G_\pi = H$ *for all* $\pi \in \hat{A}$. *If the action* α *makes* $\hat{A}$ *into a locally trivial principal* G/H*-bundle over* $\hat{A}/G$, *then the dual action* $\hat{\alpha}$ *of* $H^\perp \subset \hat{G}$ *on* $A \rtimes_\alpha G$ *is locally unitary.*

PROOF. Since the property we wish to prove is local, we claim it is enough to show that if $p\colon \hat{A} \to \hat{A}/G$ is actually a trivial G/H-bundle, then $\hat{\alpha}|_{H^\perp}$ is implemented by some $u\colon H^\perp \to U(M(A)) \to U(M(A \rtimes_\alpha G))$. For if I is the G-invariant ideal in A corresponding to an open G-invariant subset of $\hat{A}$ over which p is trivial, then $I \rtimes_\alpha G$ is an $\hat{\alpha}$-invariant ideal in $A \rtimes_\alpha G$. If $u\colon H^\perp \to U(M(I))$ implements $\hat{\alpha}|_{H^\perp}$ on $I \rtimes_\alpha G$, then shrinking I a little we can embed the image of u into $M(A \rtimes_\alpha G)$, and varying the I we see $\hat{\alpha}|_{H^\perp}$ is locally unitary.

So suppose $\pi\colon G/H \times T \to \hat{A}$ is a G-equivariant homeomorphism. For $\gamma \in H^\perp$, define $u_\gamma\colon \hat{A} \to \mathbb{T}$ by

$$u_\gamma(\pi(sH, t)) = \overline{\gamma(s)} \qquad (s \in G,\ t \in T),$$

and view u_γ as a central unitary in $M(A)$ using the Dauns-Hofmann Theorem [**24**, §4.4]. It is clear that $\gamma \mapsto u_\gamma$ is a strictly continuous homomorphism. Let v_γ be the canonical image of u_γ in $M(A \rtimes_\alpha G)$; recall that this means

$$(v_\gamma z)(s) = u_\gamma(z(s)), \quad (z v_\gamma)(s) = z(s)\alpha_s(v_\gamma) \qquad \text{for } z \in C_c(G, A).$$

Since $\pi(rH,t)\circ\alpha_s \cong \pi(s^{-1}rH,t)$, there is a unitary W such that

$$\pi(rH,t)(\alpha_s(a)) = W\pi(s^{-1}rH,t)(a)W^* \quad \text{for all } a \in A.$$

Then for $z \in C_c(G,A)$,

$$\begin{aligned}\pi(rH,t)\big((v_\gamma z v_\gamma^*)(s)\big) &= \pi(rH,t)\big(u_\gamma z(s)\alpha_s(u_\gamma^*)\big)\\ &= \overline{\gamma(r)}\,\pi(rH,t)\circ\alpha_s\big(\alpha_s^{-1}(z(s))u_\gamma^*\big)\\ &= \overline{\gamma(r)}\,W\pi(s^{-1}rH,t)\big(\alpha_s^{-1}(z(s))u_\gamma^*\big)W^*\\ &= \overline{\gamma(r)}\,\gamma(s^{-1}r)W\pi(s^{-1}rH,t)\circ\alpha_s^{-1}(z(s))W^*\\ &= \overline{\gamma(s)}\,\pi(rH,t)(z(s)) = \pi(rH,t)\big(\hat{\alpha}_\gamma(z)(s)\big).\end{aligned}$$

Thus $v_\gamma z v_\gamma^* = \hat{\alpha}_\gamma(z)$ for all $z \in C_c(G,A)$, hence for all $z \in A \rtimes_\alpha G$ by continuity. The result follows. □

PROOF OF THEOREM 2.2. We first observe that in this situation, we may replace the primitive ideal spaces in Proposition 2.1 by spectra. This is elementary in the case of $A \rtimes_\alpha H$ (Theorem 0.16), and in the case of $A \rtimes_\alpha G$ it follows by an application of Green's Mackey machine. In fact, since the G-orbits in $\hat{A}$ are closed, every irreducible representation of $A \rtimes_\alpha G$ factors through $(A/I) \rtimes_\alpha G$, where I is the ideal corresponding to some orbit $G \cdot P$ in $\operatorname{Prim} A \cong \hat{A}$. By [**12**, Theorem 2.13], $(A/I) \rtimes G$ is stably isomorphic to $(A/P) \rtimes H$. However, $\alpha|_H$ is locally unitary, and in particular pointwise unitary, so $(A/P) \rtimes H \cong C_0(\hat{H}, A/P)$. This is a continuous-trace algebra, so each irreducible representation is determined by its kernel, as we claimed.

By Theorems 0.16 and 0.17, Res: $(A \rtimes_\alpha H)^\wedge \to \Omega$ is a locally trivial $\hat{H}$-bundle for the dual action of $\hat{H}$, which is given by $\gamma \cdot (\pi \times U) = (\pi \times \gamma U)$. Since G acts on $(A \rtimes_\alpha H)^\wedge$ by $s \cdot (\pi \times U) = (s \cdot \pi) \times U$, every point has stabilizer H for this action, and furthermore, the action commutes with the action of $\hat{H}$ and Res is (G/H)-equivariant. Thus $p \circ \text{Res}$ is a locally trivial $(\hat{H} \times G/H)$-bundle over T. Dividing first by the G/H-action therefore gives a locally trivial G/H-bundle p_1: $(A \rtimes_\alpha H)^\wedge \to X$, with q_1: $X \to T$ a principal $\hat{H}$-bundle. Note that $q_1 \circ p_1 = p \circ \text{Res}$.

We can identify X with the quasi-orbit space Q of Proposition 2.1, so that by the last part of that proposition Ind induces a homeomorphism h of X onto $(A \rtimes_\alpha G)^\wedge$. By Lemma 2.3, induction is equivariant for the dual action of $\hat{H}$, so q: $(A \rtimes_\alpha G)^\wedge \to T$ may be identified with q_1 and so is a principal $\hat{H}$-bundle.

It remains to establish the last part of the theorem. As in the proof of Proposition 2.1, we have an isomorphism R: $A \rtimes_\alpha G \to C^*(G, A \rtimes_\alpha H, \mathcal{T}^H)$ given by [**11**, Proposition 1], and it is easy to see that R carries the dual action of $H^\perp \cong (G/H)^\wedge$ on $A \rtimes_\alpha G$ to the action of $(G/H)^\wedge$ described in [**11**, p. 235]. Thus by [**11**, Corollary 31], $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} H^\perp$ is strongly Morita equivalent to $A \rtimes_\alpha H$, and in particular their spectra are homeomorphic. (This gives part of the Takai duality that reverses the two sides in the diagram of the theorem.) This equivalence is demonstrated in [**11**, pp. 235–236] by exhibiting an isomorphism

$$S\colon C^*\big(G, A \rtimes_\alpha H, \mathcal{T}^H\big) \to \big(C_0(G/H) \otimes (A \rtimes_\alpha H)\big) \rtimes G,$$

where the algebra on the right is the imprimitivity algebra for inducing representations from (A, H) to (A, G). Examining the constructions of R and S shows that the homeomorphism

$$(A \rtimes_\alpha H)^\wedge \to ((A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} H^\perp)^\wedge$$

sends a representation $\pi \times U$ to $(\operatorname{Ind}(\pi \times U)) \times V$, where V is a natural action of $H^\perp$. Thus the diagram

$$\begin{array}{ccc} (A \rtimes_\alpha H)^\wedge & \xrightarrow{\cong} & ((A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} H^\perp)^\wedge \\ & \text{Ind} \searrow \qquad \swarrow \text{restriction} & \\ & (A \rtimes_\alpha G)^\wedge & \end{array}$$

commutes. By Lemma 2.4, $\hat{\alpha}|_{H^\perp}$ is locally unitary, and thus by Theorem 0.17, $A \rtimes_\alpha H$ is strongly Morita equivalent to $\operatorname{Ind}^*(A \rtimes_\alpha G)$, and isomorphic to $\operatorname{Res}^* A$.

Finally, we have seen that if (A, G, α, H) is replaced by $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha}, H^\perp)$ then all the hypotheses of the theorem are again satisfied, and we get back the mirror image of our original diagram via Takai duality [**11**, Corollary 31]. □

COROLLARY 2.5. *Let A be a C^*-algebra with Hausdorff spectrum Ω, G a locally compact abelian group, and H a closed subgroup of G. Suppose α: $G \to \operatorname{Aut} A$ makes Ω into a locally trivial principal G/H-bundle and that $\alpha|_H$ is locally unitary. Let* Ind, Res, *p, q be as in Proposition* 2.1 *and Theorem* 2.2. *Then*

(1) $\zeta(\hat{\alpha}|_{H^\perp}) = q^*([p])$, $\zeta(\alpha|_H) = p^*([q])$;

(2) *if A has continuous trace, so does $A \rtimes_\alpha G$, and its Dixmier-Douady class satisfies* $\operatorname{Ind}^* \delta(A \rtimes_\alpha G) = \operatorname{Res}^* \delta(A)$.

PROOF. (1) follows immediately from the theorem since the actions of $\hat{H}$ and of G/H on $(A \rtimes_\alpha H)^\wedge$ commute. (2) follows from the last part of the theorem, Lemma 1.2 and Theorem 0.17. □

REMARK. Part (2) of Corollary 2.5 is not usually sufficient to determine $\delta(A \rtimes_\alpha G)$ uniquely. In the important special case when $G = \mathbb{R}$ and $H \cong \mathbb{Z}$, we shall give a more precise result in §4 below.

3. Examples and applications.

(a) *Induced C^*-algebras*. Let D be a C^*-algebra with spectrum T, and let H be a closed subgroup of a locally compact group G. Given an action θ of H on D we can form the *induced* C^*-algebra

$$A = \operatorname{Ind}_H^G D = \{ f \in C(G, D) \colon f(sh) = \theta_h^{-1}(f(s)) \text{ for } s \in G,\ h \in H, \text{ and } \|f\| \in C_0(G/H) \}.$$

(Note that since θ_h is isometric, $\|f(s)\|$ only depends on the coset sH.) There is a natural *induced action* α of G on A given by left translation: $\alpha_g(f)(s) = f(g^{-1}s)$. A fundamental fact about such actions, due essentially to P. Green but by now folklore (cf. [**29**, Situation 4]), is that $A \rtimes_\alpha G$ is strongly Morita equivalent to $D \rtimes_\theta H$. For the case of most interest to us, we have given a more precise statement and a formal proof as Lemma 3.1.

Induced actions provide interesting and tractable examples of the theory of §2. If T is Hausdorff, G is abelian and θ is locally unitary, it turns out $\alpha|_H$ is also locally unitary, and $\hat{A}$ is isomorphic as a (G/H)-space to $(G/H) \times T$. In these examples the bundle q: $(A \rtimes_\alpha G)^\wedge \to T$ can be nontrivial, and if D has continuous trace, $\delta(A \rtimes_\alpha G) = q^*\delta(D)$. This result when $G = \mathbb{R}$ will be used and amplified in §4 below.

LEMMA 3.1. *Let G be a locally compact group, H a closed subgroup and θ an action of H on a C^*-algebra D which fixes* Prim D *(pointwise). Let α be the induced action of G on $A = \mathrm{Ind}_H^G D$. Then $A \rtimes_\alpha G$ is strongly Morita equivalent to $D \rtimes_\theta H$, and the corresponding homeomorphism of spectra is given by*

$$\pi \times U \in (D \rtimes_\theta H)^\wedge \mapsto \mathrm{Ind}_H^G((\pi \circ \varepsilon_e) \times U) \in (A \rtimes_\alpha G)^\wedge,$$

where ε_e is the homomorphism $a \mapsto a(e)$ of A onto D.

PROOF We define M: $G \times \hat{D} \to \hat{A}$ by $M(g, \pi)(a) = \pi(a(g))$. The proof of [**28**, Lemma 2.6] shows that M induces a bijection of $G/H \times \hat{D}$ onto $\hat{A}$, and the proof of [**28**, Proposition 3.1] shows that this bijection is a homeomorphism. (In the language of [**28**], $\mathrm{Ind}_H^G D$ is $GC(G, D)^\beta$ for the action β of H defined by $\beta_h(f)(s) = \theta_h(f(sh))$. The overriding assumption in [**28**] that D have Hausdorff spectrum is not used here, and the change to right translation by H clearly does not matter.) This gives us a homeomorphism φ of $G/H \times$ Prim D onto Prim A defined (for $s \in G$, $J \in$ Prim D) by

$$\varphi(sH, J) = \{a \in A: a(s) \in J\}.$$

Clearly φ is G-equivariant, so the composition of φ^{-1} with the projection onto G/H gives a G-equivariant map π of Prim A onto G/H. If we write $I = \ker \pi^{-1}\{eH\}$, then by [**11**, Theorem 17] $A \rtimes_\alpha G$ is strongly Morita equivalent to $A \rtimes_\alpha H/I \rtimes_\alpha H$. Further, the imprimitivity bimodule X implementing this equivalence is a quotient of the module which induces representations of $A \rtimes_\alpha H$ to $A \rtimes_\alpha G$, so the corresponding homeomorphism of spectra is given by this induction. It is easy to verify that I is the kernel of ε_e, and that this map intertwines $\alpha|_H$ with θ, so the result follows from the isomorphism

$$(A \rtimes_\alpha H)/(I \rtimes_\alpha H) \cong (A/I) \rtimes H \cong D \rtimes_\theta H. \quad \square$$

PROPOSITION 3.2. *Let G be a locally compact abelian group, H a closed subgroup, D a C^*-algebra with Hausdorff spectrum T and θ: $H \to \mathrm{Aut}\, D$ a locally unitary automorphism group. Denote by α the action of G by translation on the induced C^*-algebra $A =$ Ind D. Then $\alpha|_H$ is locally unitary, and $\hat{A}$ is G/H-isomorphic to the product space $(G/H) \times T$. The algebra $A \rtimes_\alpha G$ is strongly Morita equivalent to $D \rtimes_\theta H$, and the $\hat{H}$-bundle q: $(A \rtimes_\alpha G)^\wedge \to T$ of* §2 *is isomorphic to the bundle q_1: $(D \rtimes_\theta H)^\wedge \to T$ given by restriction.*

PROOF. Suppose u: $H \to M(D)$ implements θ over an open set V in T, so that for $\rho \in V$, $h \mapsto \rho(u_h)$ is a representation of H which implements θ in the representation ρ. We then have $\rho(\theta_h(u_k)) = \rho(u_k)$ for $\rho \in V$ and h, $k \in H$. If f is a continuous function which is identically 1 on an open set U and 0 off V, then $w_k = fu_k$ implements θ over U and is invariant under θ. Pointwise multiplication by w_k therefore defines a multiplier v_k of A. For $\pi \in U$, $s \in G$ let $M(s, \pi)$ denote the representation $a \mapsto \pi(a(s))$. Then

$$M(s,\pi)(\alpha_h(a)) = \pi(a(h^{-1}s)) = \pi(\theta_h(a(s))) \quad \text{(since } G \text{ is abelian)}$$
$$= \pi(w_h a(s) w_h^*) = M(s,\pi)(v_h a v_h^*).$$

But $\{M(s,\pi)\colon s \in G, \pi \in U\}$ is open in $\hat{A}$ (see the proof of Lemma 3.1), so this proves that $\alpha|_H$ is locally unitary. The statement about $\hat{A}$ was also established while proving Lemma 3.1, and by that result $A \rtimes_\alpha G$ is Morita equivalent to $D \rtimes_\theta H$. The dual action of $\hat{H}$ on $(D \rtimes_\theta H)^\wedge$ is given by $\gamma \cdot (\pi \times U) = \pi \times (\gamma \otimes U)$, so the description of $(A \rtimes_\alpha G)^\wedge$ follows from the last part of Lemma 3.1 and from Lemma 2.3. □

REMARK. Since any $\hat{H}$-bundle over T can be realized as $(D \rtimes_\theta H)^\wedge$ (provided T is paracompact) by Theorem 0.16(b), this shows that q could be any $\hat{H}$-bundle. It follows from Theorem 0.17 that $A \rtimes_\alpha G$ is strongly Morita equivalent to the pull-back q^*D.

When the algebra D in Proposition 3.2 has continuous trace, so do A, $A \rtimes_\alpha H$ and $A \rtimes_\alpha G$. It follows from the proposition that $\delta(A \rtimes_\alpha G) = q^*\delta(D)$, and in many cases we can also calculate $\delta(A)$ and $\delta(A \rtimes_\alpha H)$. We suppose in addition that $r\colon G \to G/H$ has local cross-sections (this is automatic if H is a Lie group, by the corollary to Theorem 4.1 of **[23]**).

LEMMA 3.3. *Suppose G is a locally compact abelian group, $p\colon \Omega \to T_1$ is a (locally trivial) principal G-bundle, B is a continuous-trace C^*-algebra with spectrum T_2 and $\beta\colon G \to \operatorname{Aut} B$ is a locally unitary automorphism group. Also assume T_1, T_2 paracompact. Then if $\pi_i\colon T_1 \times T_2 \to T_i$ $(i = 1,2)$ are the projections and $\alpha_t(f)(x) = \beta_t(f(t^{-1}x))$, we have*

$$\delta(GC(\Omega, B)^\alpha) = \pi_2^*(\delta(B)) + \langle \pi_2^*(\zeta(\beta)), \pi_1^*[p]\rangle$$

(with notation as in §1*).*

PROOF. We could proceed as in the proof of Theorem 1.5, but it will be faster merely to quote the calculation done there. Let $B_1 = C_0(T_1, B)$, $p_1 = p \times \mathrm{id}\colon \Omega \times T_2 \to T_1 \times T_2$, and $\Omega_1 = \Omega \times T_2$. Define $\gamma\colon G \to \operatorname{Aut} B_1$ by $\gamma_g(f)(t) = \beta_g(f(t))$. Then if we view B as the algebra of sections of a field over T_2,

$$\Phi(f)(x)(t_2) = f(x, t_2)(p(x), t_2)$$

gives an isomorphism of a quotient of $GC(\Omega, B_1)^{\alpha_1}$ onto $GC(\Omega, B)^\alpha$. The result follows from the calculation in the proof of Theorem 1.5. □

PROPOSITION 3.4. *Let G be a locally compact abelian group, H a closed subgroup, $\theta\colon H \to \operatorname{Aut}(D)$ a locally unitary action on a continuous-trace algebra with paracompact spectrum T, $(A, \alpha) = \operatorname{Ind}_H^G(D, \theta)$. If $r\colon G \to G/H$ is locally trivial and $\pi_1\colon G/H \times T \to G/H$, $\pi_2\colon G/H \times T \to T$ are the two projections, then*

$$\begin{cases} \delta(A) = \pi_2^*(\delta(D)) - \langle \pi_2^*(\zeta(\theta)), \pi_1^*[r]\rangle, \\ \delta(A \rtimes_\alpha G) = q^*\delta(D), \\ \delta(A \rtimes_\alpha H) = (q^*\pi_2)^* \circ q^*\delta(D). \end{cases}$$

Here $q\colon (A \rtimes_\alpha G)^\wedge \cong (D \rtimes_\theta H)^\wedge \to T$ by Proposition 3.2 has $[q] = \zeta(\theta)$.

PROOF. Since G is abelian, $A = GC(G, D)^{\gamma}$, where $\gamma_s(f)(t) = \theta_s^{-1}(f(s^{-1}t))$, and $\zeta(\theta^{-1}) = -\zeta(\theta)$. The formula for $\delta(A)$ thus follows from Lemma 3.3, whereas the formula for $\delta(A \rtimes_\alpha G)$ follows from Proposition 3.2. Now $\hat{A}$ is (G/H)-isomorphic to $G/H \times T$, and by Theorem 2.2, $(A \rtimes_\alpha H)^\wedge$ is $(G/H) \times \hat{H}$-isomorphic to $(G/H) \times (D \rtimes_\theta H)^\wedge$. Since $A \rtimes_\alpha H$ is isomorphic to the pull-back of A under $\pi_2^* q$, we have

$$\begin{aligned}\delta(A \rtimes_\alpha H)^\wedge &= (\pi_2^* q)^* \left[\pi^*(\delta(D)) - \langle \pi_2^*(\zeta(\theta)), \pi_1^*[r]\rangle\right] \\ &= (q^*\pi_2)^* \circ q^*\delta(D) \in H^3\left(G/H \times (D \rtimes_\theta H)^\wedge, \mathbb{Z}\right),\end{aligned}$$

since

$$(\pi_2^* q)^* \left[\pi_2^*(\zeta(\theta))\right] = (\pi_2^* q)^* \left[\pi_2^* q\right] = 0.$$

(The pull-back of any bundle along itself is canonically trivial.) □

COROLLARY 3.5. *With notation as in Proposition* 3.4, *if* $G = \mathbb{R}$ *and* $H = \mathbb{Z}$ *(so that both* G/H *and* $\hat{H}$ *may be identified with* $\mathbb{T}$, *and* $[p]$ *and* $\zeta(\theta)$ *may be viewed as elements of* $H^2(T, \mathbb{Z})$*), we have a commutative diagram of* $\mathbb{T}$*-bundles*

$$\begin{array}{ccccc} & & \mathbb{T} \times X & & \\ & q^*p \swarrow & & \searrow p^*q & \\ X & & & & \mathbb{T} \times T \\ & q \searrow & & \swarrow p = \pi_2 & \\ & & T & & \end{array}$$

where $X = (D \rtimes_\theta \mathbb{Z})^\wedge$. *Furthermore, if* 1 *and* z *are the standard generators of* $H^0(S^1, \mathbb{Z})$ *and* $H^1(S^1, \mathbb{Z})$, *we have*

$$\begin{cases} [q] = \zeta(\theta), \qquad [p] = 0, \\ \delta(A) = 1 \times \delta(D) + z \times \zeta(\theta), \qquad \delta(A \rtimes_\alpha G) = q^*\delta(D), \\ \delta(A \rtimes_\alpha H) = 1 \times q^*\delta(D). \end{cases}$$

PROOF. We need only make explicit all the terms in the formulae of Proposition 3.4. It is standard that $r\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the covering corresponding to $z \in H^1(S^1, \mathbb{Z})$. Since $\langle \cdot, \cdot \rangle$ was defined in terms of the cup product $H^1(\cdot, \mathbb{T}) \times H^1(\cdot, \mathbb{Z}) \to H^2(\cdot, \mathbb{T})$, which is *anti*commutative,

$$\langle \pi_2^*(\zeta(\theta)), \pi_1^*[r]\rangle = \langle 1 \times \zeta(\theta), \pi_1^*(z)\rangle = -\pi_1^*(z) \cup (1 \times \zeta(\theta)) = -z \times \zeta(\theta),$$

and the result follows. □

REMARK. Corollary 3.5 will be greatly generalized in Theorem 4.12 below, but the proof of the more general formula there depends on knowledge of this special case.

(b) *Actions on certain twisted transformation group* C^**-algebras.* We shall now look at a class of examples where the Dixmier-Douady class $\delta(A)$ and the bundles, p, q are all nontrivial, generally with torsion. Let H be a finitely generated discrete subgroup of a locally compact abelian group G, and suppose φ is a homomorphism of H into a finite abelian group K (actually, it is enough that H goes into the center of a finite group). We suppose there are commuting actions of G and K on a space X which are consistent with φ (i.e., the actions of $h \in H \subseteq G$ and of $\varphi(h) \in K$

coincide), and also that K acts freely on X and G/H acts freely on X/K. (We shall give a typical example shortly.) Let A be the twisted crossed product $C^*(K, X, \omega)$ for some multiplier ω on K; by definition, A is generated by formal products fu_k, $f \in C_0(X)$, $k \in K$, where the u_k's are unitary elements of $M(A)$ and the generators are to satisfy the multiplication rules

$$(*) \qquad \begin{cases} u_k f u_k^* \in C_0(X), & (u_k f u_k^*)(x) = f(k^{-1}x), \\ u_k u_l = \omega(k,l) u_{kl}. \end{cases}$$

A theorem of A. Wassermann ([**35**, Theorem 5]; see also [**28**, §4]) asserts that A is a continuous-trace algebra with spectrum $\Omega = X/K$, whose Dixmier-Douady class is the pull-back of

$$[\omega] \in H^2(K, \mathbb{T}) \cong H^3(K, \mathbb{Z}) \cong H^3(BK, \mathbb{Z})$$

under the classifying map $\Omega \to BK$ for the K-bundle $X \to \Omega$. We may define an action α of G on A by

$$\alpha_g(f)(x) = f(g^{-1}x), \qquad \alpha_g(u_k) = u_k,$$

since the relations $(*)$ are preserved.

We want $\alpha|_H$ to be locally unitary. Since H is finitely generated, this will happen whenever it is pointwise unitary [**27**, Proposition 1.1]. We shall use the following description of the spectrum of A: for $x \in X$, define $\rho_x = \pi_x \times U \in \hat{A}$, where π_x and U act on $L^2(K)$ according to

$$\begin{cases} (\pi_x(f)\xi)(l) = f(lx)\xi(l), \\ (U(k)\xi)(l) = \omega(k, k^{-1}l)\xi(k^{-1}l). \end{cases}$$

In fact, ρ_x is the ω-representation of $(C_0(X), K)$ induced from evaluation at x. It is straightforward to check that the unitary W_l on $L^2(K)$ defined by

$$(W_l\xi)(k) = \overline{\omega(k,l)}\,\xi(kl)$$

satisfies

$$(**) \qquad W_l \rho_x W_l^* = \rho_{lx},$$

and the map $x \mapsto \rho_x$ gives rise to a homeomorphism of X/K onto $\hat{A}$. One checks that $l \mapsto W_l$ is an $\bar{\omega}$-representation of K, and that

$$\rho_x \circ \alpha_h^{-1}(a) = \rho_{\varphi(h)x}(a) = W_{\varphi(h)}\rho_x(a)W_{\varphi(h)}^* \quad \text{for } a \in A,$$

so $\alpha|_H$ will be pointwise unitary precisely when $\omega \circ (\varphi \times \varphi)$ is a trivial multiplier on H. We therefore suppose ω is trivial on $\varphi(H)$, and adjust it so it is identically 1 there. Then we can define a continuous map ψ of $\hat{H} \times X$ onto $(A \rtimes_\alpha H)^\wedge$ by

$$\psi(\gamma, x) = \rho_x \times [\gamma \otimes (W \circ \varphi)].$$

It follows from $(**)$ that

$$\begin{aligned} \rho_{kx} \times [\gamma \otimes (W \circ \varphi)] &= (W_k \rho_x W_k^*) \times [\gamma \otimes (W \circ \varphi)] \\ &= W_k(\rho_x \times [\gamma \otimes \Xi(k) \otimes (W \circ \varphi)])W_k^*, \end{aligned}$$

where $\Xi\colon K \to \hat{H}$ is given by

$$\Xi(k)(h) = \omega(k, \varphi(h))\,\overline{\omega(\varphi(h), k)}\,.$$

Thus ψ gives a homeomorphism of the fiber product $\hat{H} \times_K X$ (defined via Ξ) onto $(A \rtimes_\alpha H)^\wedge$, and ψ is in fact an isomorphism of principal $\hat{H}$-bundles over X/K. The locally unitary group $\alpha|_H$ will be unitary precisely when this $\hat{H}$-bundle is trivial, by Theorem 0.16.

EXAMPLE 3.6. We shall now present a concrete example where we can calculate all these invariants. (It will be clear that the parameters involved can be altered to produce a large stock of similar examples.) Let $X = S^5 \times S^5 \subset \mathbb{C}^3 \times \mathbb{C}^3$, $K = \mathbb{Z}_2 \times \mathbb{Z}_2$, $G = \mathbb{R}$, and $H = \mathbb{Z}$. We let K and G act on X by

$$(m,n)\cdot(z,w) = ((-1)^m z, (-1)^n w) \quad \text{for } (m,n) \in \mathbb{Z} \times \mathbb{Z} \ (\operatorname{mod} 2\mathbb{Z} \times 2\mathbb{Z}),$$

$$s \cdot (z,w) = (e^{\pi i s} z, w) \quad \text{for } s \in \mathbb{R},$$

and define the multiplier ω by

$$\omega((m,n),(s,t)) = (-1)^{mt}.$$

Note that the actions of H and K are compatible with the obvious homomorphism φ defined by

$$\varphi(m) = (m + 2\mathbb{Z}, 0), \qquad m \in \mathbb{Z} \subset \mathbb{R}.$$

$\Omega = X/K$ is homeomorphic to $\mathbb{RP}^5 \times \mathbb{RP}^5$ and ω defines a nonzero class in $H^2(K, \mathbb{T}) \cong \mathbb{Z}_2$. A classifying space for $\mathbb{Z}_2$ is $\mathbb{RP}^\infty$, so we can take $BK = \mathbb{RP}^\infty \times \mathbb{RP}^\infty$. For large k the embeddings $\mathbb{RP}^k \times \mathbb{RP}^k \to BK$ induce isomorphisms on low-dimensional cohomology, so the map

$$H^2(K, \mathbb{T}) \cong H^3(K, \mathbb{Z}) \cong H^3(BK, \mathbb{Z}) \to H^3(\mathbb{RP}^5 \times \mathbb{RP}^5, \mathbb{Z})$$

is an isomorphism, and $\delta(A) \neq 0$ since ω is nontrivial. The multiplier ω is identically 1 on $\varphi(H) = \mathbb{Z}_2 \times \{0\}$, so $\alpha|_{\mathbb{Z}}$ is locally unitary. It remains to identify the bundles p and q of Theorem 2.2.

The homomorphism $\Xi\colon K \to \mathbb{T} = \hat{\mathbb{Z}}$ is given by $\Xi(m,n) = (-1)^n$ ($(m,n) \in \mathbb{Z} \times \mathbb{Z} \bmod 2\mathbb{Z} \times 2\mathbb{Z}$), so $\mathbb{T} \times_K X$ is therefore isomorphic to $\mathbb{RP}^5 \times E$, where $E \to \mathbb{RP}^5$ is the $\mathbb{T}$-bundle corresponding to the generator of $H^2(\mathbb{RP}^5, \mathbb{Z}) \cong \mathbb{Z}_2$. ($E$ comes from the map $H^1(\mathbb{RP}^5, \mathbb{Z}_2) \to H^1(\mathbb{RP}^5, \underline{\mathbb{T}})$ coming from the inclusion $\mathbb{Z}_2 \cong \{\pm 1\} \hookrightarrow \mathbb{T}$. The image is nontrivial as one can see from the exact cohomology sequence

$$\cdots \to H^0(\mathbb{RP}^5, \underline{\mathbb{T}}) \xrightarrow{a} H^0(\mathbb{RP}^5, \underline{\mathbb{T}}) \xrightarrow{b} H^1(\mathbb{RP}^5, \mathbb{Z}_2) \xrightarrow{c} H^1(\mathbb{RP}^5, \underline{\mathbb{T}}) \to \cdots$$

associated to the squaring map $\mathbb{T} \to \mathbb{T}$. Since $H^1(\mathbb{RP}^5, \mathbb{Z}) = 0$, every continuous function $\mathbb{RP}^5 \to \mathbb{T}$ has a square root; hence a is surjective and c is injective.) One can see now that the diagram of Theorem 2.2 becomes in this case

$$\begin{array}{ccccc} & & \mathbb{RP}^5 \times E & & \\ & q^*p \swarrow & & \searrow p^*q & \\ \mathbb{CP}^2 \times E & & & & \mathbb{RP}^5 \times \mathbb{RP}^5. \\ & q \searrow & & \swarrow p & \\ & & T = \mathbb{CP}^2 \times \mathbb{RP}^5 & & \end{array}$$

Formulae for the various Dixmier-Douady classes may be deduced from the fact that

$$\delta(A) \in H^3(\mathbb{RP}^5 \times \mathbb{RP}^5, \mathbb{Z}) \cong \operatorname{Tor}(H^2(\mathbb{RP}^5, \mathbb{Z}), H^2(\mathbb{RP}^5, \mathbb{Z})) \cong \mathbb{Z}_2$$

is the nontrivial element of this group and from the formulae to be given in Theorem 4.12 below. □

(c) *Actions on continuous-trace algebras over the* 3-*sphere*. We shall show, as a concrete example of the theory of the next section, how to construct an action of $\mathbb{R}$ on any stable continuous-trace algebra A with spectrum S^3 so that each point in $\hat{A}$ has stabilizer $\mathbb{Z}$ and the quotient map $\hat{A} \to \hat{A}/\mathbb{R}$ is the Hopf fibration $S^3 \to S^2$. At first sight, this is somewhat surprising, since there could be no corresponding action of $\mathbb{T}$ unless $\delta(A) = 0$ (see Remark 4.7 below). A more general existence theorem for $\mathbb{R}$-actions will be given later as Theorem 4.8; however, the present construction is more explicit.

EXAMPLE 3.7. Let $\Omega = S^3$, realized as the unit sphere in $\mathbb{C}^2$, and let $t \in \mathbb{R}$ act by scalar multiplication by $e^{2\pi i t}$, so that $\Omega \to \Omega/\mathbb{R}$ is the Hopf fibration $S^3 \to S^2$. Let

$$\Omega_+ = \{(w,z) \in \Omega: |w| \geqslant |z|\}, \quad \Omega_- = \{(w,z) \in \Omega: |w| \leqslant |z|\},$$

so that $\Omega = \Omega_+ \cup \Omega_-$, $\Omega_\pm$ are invariant under $\mathbb{R}$, and

$$\Omega_+ \cap \Omega_- = \{(w,z) \in \Omega: |w| = |z| = \sqrt{2}\}$$

is homeomorphic to the torus T^2. Given a continuous map $\varphi: T^2 \to PU(\mathscr{H}) \cong \operatorname{Aut} \mathscr{K}(\mathscr{H})$, where $\mathscr{H}$ is an infinite-dimensional separable Hilbert space, set

$$A_\varphi = \{(f,g): f \in C(\Omega_+, \mathscr{K}(\mathscr{H})),\ g \in C(\Omega_-, \mathscr{K}(\mathscr{H})),$$
$$\text{and } f(x) = \varphi(x)(g(x)) \text{ for } x \in T^2\}.$$

It is not hard to see that every stable continuous-trace algebra with spectrum Ω is isomorphic to one of these. (In fact, the homotopy class of φ in $[T^2, PU] \cong H^2(T^2, \mathbb{Z})$ may be identified with $\delta(A_\varphi) \in H^3(\Omega, \mathbb{Z})$ via the boundary map in the Mayer-Vietoris sequence for (Ω_+, Ω_-) [**30**, §6].) We shall define an action of $\mathbb{R}$ on A_φ by setting

$$\begin{cases} \alpha_t(f,g) = (f_t, g_t), \quad \text{where} \\ f_t(w,z) = f(e^{-2\pi i t}w, e^{-2\pi i t}z), \\ g_t(w,z) = \psi(t,w,z)\big(g(e^{-2\pi i t}w, e^{-2\pi i t}z)\big), \end{cases}$$

for some map $\psi: \mathbb{R} \times \Omega_- \to PU$ such that

(1) $\quad \psi(t,w,z) = \varphi(w,z)^{-1}\varphi(e^{-2\pi i t}w, e^{-2\pi i t}z) \quad$ for $(w,z) \in T^2$,

(2) $\quad \psi(r+s,w,z) = \psi(r,w,z)\psi(s, e^{-2\pi i r}w, e^{-2\pi i r}z) \quad$ for $(r,w,z) \in \mathbb{R} \times \Omega_-$.

Condition (1) ensures that each α_t is an automorphism of A_φ, and condition (2) that $t \mapsto \alpha_t$ is a homomorphism.

It remains to show that such cocycles ψ exist. Let

$$Y = \left\{\left(\zeta/\sqrt{1+|\zeta|^2}, 1/\sqrt{1+|\zeta|^2}\right): |\zeta| < 1\right\},$$

which meets each $\mathbb{R}$-orbit in $\Omega_- \setminus T^2$ exactly once, and let $X = Y \cup T^2$. We define ψ on $\mathbb{R} \times T^2$ by condition (1). The restriction of ψ to $[0,1] \times T^2$ is null-homotopic (take $\psi_s(t,x) = \psi(st,x)$), so by the homotopy extension theorem we can extend ψ

(viewed itself as a homotopy of functions $\psi(t,\cdot)\colon T^2 \to PU$) over the 2-cell Y to a continuous function (which we will again call ψ) $[0,1] \times X \to PU$ with $\psi(0,\cdot) \equiv 1 \in PU$. Now extend ψ to a map $\mathbb{R} \times X \to PU$ by

$$\psi(t,x) = \psi(1,x)^{[t]}\psi(t-[t],x), \tag{3}$$

where $[t]$ denotes the greatest integer $\leqslant t$. It is easy to check that this is a well-defined continuous function.

We shall now extend ψ to all of $\mathbb{R} \times \Omega_-$ so that the cocycle condition (2) is satisfied. Let

$$\psi(s,w,e^{-2\pi it}|z|) = \psi(t,e^{2\pi it}w,|z|)^{-1}\psi(t+s,e^{2\pi it}w,|z|).$$

This makes sense since the right-hand side only involves $\psi|_{\mathbb{R}\times X}$, and since (3) shows that replacing t by $t+n$ ($n \in \mathbb{Z}$) does not change the formula. If we write an arbitrary z as $e^{-2\pi it}|z|$, then

$$\begin{aligned}
\psi(r+s,w,z) &= \psi(t,e^{2\pi it}w,|z|)^{-1}\psi(t+r+s,e^{2\pi it}w,|z|)\\
&= \left[\psi(t,e^{2\pi it}w,|z|)^{-1}\psi(t+r,e^{2\pi it}w,|z|)\right]\\
&\quad\cdot\left[\psi(t+r,e^{2\pi it}w,|z|)^{-1}\psi(t+r+s,e^{2\pi it}w,|z|)\right]\\
&= \psi(r,w,z)\left[\psi(t+r,e^{2\pi i(t+r)}e^{-2\pi ir}w,|z|)^{-1}\right.\\
&\qquad\qquad\left.\cdot\,\psi((t+r)+s,e^{2\pi i(t+r)}e^{-2\pi ir}w,|z|)\right]\\
&= \psi(r,w,z)\psi(s,e^{-2\pi ir}w,e^{-2\pi ir}(e^{-2\pi it}|z|))
\end{aligned}$$

as required. □

4. Actions of $\mathbb{R}$. Among locally compact groups, the real line $\mathbb{R}$ is unique both in the frequency with which it occurs in applications and in special properties not shared by other continuous groups. For these reasons, we devote this section to the detailed study of actions of $\mathbb{R}$ on C^*-algebras in general and continuous-trace algebras in particular. Then we discuss some applications to the study of C^*-algebras of solvable Lie groups.

One special feature of the real line is that it turns out to be feasible in some cases to classify all the actions of $\mathbb{R}$ on a given C^*-algebra, up to exterior equivalence. The key tool is the following vanishing theorem for Moore cohomology, which effectively answers a question raised in [**14**, Remark 2.10]. For similar results regarding groups other than $\mathbb{R}$, see [**32**].

THEOREM 4.1. *Let X be a compact metrizable space (or more generally, a second-countable locally compact space with $H^0(X,\mathbb{Z})$ and $H^1(X,\mathbb{Z})$ countable—recall Lemma 0.3), equipped with a continuous action of $\mathbb{R}$ by homeomorphisms. Give $C(X,\mathbb{T})$ the topology of uniform convergence on compacta and the action of $\mathbb{R}$ by translation of functions. Then the Moore cohomology groups $H^n(\mathbb{R},C(X,\mathbb{T}))$ vanish for $n \geqslant 2$.*

PROOF. We have the short exact sequences of $\mathbb{R}$-modules:

$$(*) \qquad 1 \to C(X,\mathbb{T})_0 \to C(X,\mathbb{T}) \to H^1(X,\mathbb{Z}) \to 1$$

and

$$(**) \qquad 1 \to H^0(X,\mathbb{Z}) \to C(X,\mathbb{R}) \to C(X,\mathbb{T})_0 \to 1,$$

where $(**)$ arises from the long exact sheaf cohomology sequence associated to the exact sequence $0 \to \mathbb{Z} \to \underset{\sim}{\mathbb{R}} \to \underset{\sim}{\mathbb{T}} \to 0$ of sheaves over X. Here $C(X,\mathbb{T})_0$ denotes the connected component of the identity in $C(X,\mathbb{T})$, and $(*)$ and $(**)$ are in fact topological exact sequences of Polish modules when $H^0(X,\mathbb{Z})$ and $H^1(X,\mathbb{Z})$ are given the discrete topology, by the same argument used in the proof of Theorem 0.8.

Now since $H^0(X,\mathbb{Z})$ and $H^1(X,\mathbb{Z})$ are countable and discrete, $\mathbb{R}$ acts trivially on them. But for A a countable discrete $\mathbb{R}$-module, Theorem 4 of **[37]** shows $H^n(\mathbb{R},A) \cong H^n(B\mathbb{R},A) = 0$ for $n \geqslant 1$. Using this together with the long exact cohomology sequences coming from $(*)$ and $(**)$ gives

$$H^n(\mathbb{R},C(X,\mathbb{T})_0) \cong H^n(\mathbb{R},C(X,\mathbb{T})) \quad \text{for } n \geqslant 2$$

and

$$H^n(\mathbb{R},C(X,\mathbb{R})) \cong H^n(\mathbb{R},C(X,\mathbb{T})_0) \quad \text{for } n \geqslant 1.$$

Hence, to prove our theorem, it suffices to show $H^n(\mathbb{R},C(X,\mathbb{R})) = 0$ for $n \geqslant 2$. Since $C(X,\mathbb{R})$ is a topological vector space (in fact a Fréchet space), we may work with the "continuous cochain" cohomology theory by **[37**, Theorem 3]. Then by **[13**, Corollaire III.7.5], a form of the Van Est Theorem applies and gives

$$H^n(\mathbb{R},C(X,\mathbb{R})) \cong H^n_{\mathrm{Lie}}(\mathbb{R},C(X,\mathbb{R})_\infty),$$

where $C(X,\mathbb{R})_\infty$ is the space of functions on X which are C^∞ in the direction of the flow, and on the right-hand side we use Lie algebra cohomology. But for any Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module V, $H^n(\mathfrak{g},V) = 0$ for $n > \dim \mathfrak{g}$, so this proves the theorem. □

COROLLARY 4.2. *Let A be a unital, separable C^*-algebra all of whose bounded derivations are inner. Let α,β: $\mathbb{R} \to \mathrm{Aut}(A)$ be two one-parameter automorphism groups of A such that $\|\alpha_t - \beta_t\| < 2$ for all t with $|t| < C$ for some C. Then α is exterior equivalent to β.*

PROOF. Use the same proof as for Theorem 2.9 of **[14]**, substituting Theorem 4.1 for **[14**, Theorem 2.6]. □

COROLLARY 4.3. *Let X be a second-countable locally compact space with $H^n(X,\mathbb{Z})$ countable for $n \leqslant 2$ (it suffices for X to have the homotopy type of a compact metric space), and let A be a separable continuous-trace algebra with spectrum X. Then given any continuous action of $\mathbb{R}$ on X, there is at most one action (up to exterior equivalence) of $\mathbb{R}$ on A inducing this action on the spectrum.*

PROOF. Combine Corollary 0.13 with Theorem 4.1. □

COROLLARY 4.4. *Let X be as in Corollary 4.3 and let A be a trivial continuous-trace algebra over X, i.e., $C_0(X, M_k)$ for some finite k or $C_0(X, \mathcal{K})$. Then given any continuous action of $\mathbb{R}$ on X, there is exactly one action of $\mathbb{R}$ on A (up to exterior equivalence) inducing this action on the spectrum.*

PROOF. All we need besides Corollary 4.3 is the existence of liftings of $\mathbb{R}$-actions on $C_0(X)$ to $\mathbb{R}$-actions on A, which is obvious. □

Curiously, Corollary 4.4 may be combined with Corollary 3.5 to give an interesting result on the transformation group C^*-algebras which is a little hard to see directly. In particular, we obtain a negative answer to a question of P. Muhly and D. Williams [**22**], who asked if $C_0(X) \rtimes \mathbb{R}$ always has vanishing Dixmier-Douady invariant when $\mathbb{R}$ acts on a space X with constant (or continuously varying) isotropy groups, so that the crossed product has continuous trace by [**39**, Theorem 5.1].

PROPOSITION 4.5. *Let X be a second-countable locally compact space with $H^n(X, \mathbb{Z})$ countable for $n \leqslant 2$, equipped with an action β of $\mathbb{R}$ for which every point has stability group $\mathbb{Z}$. Let $T = X/\mathbb{R}$, so that $p\colon X \to T$ is a principal $\mathbb{T} = \mathbb{R}/\mathbb{Z}$-bundle. Then $C_0(X) \rtimes_\beta \mathbb{R}$ is a continuous-trace algebra with spectrum $\mathbb{T} \times T$ and Dixmier-Douady class $z \times [p]$, where $[p] \in H^2(T, \mathbb{Z})$ and z is the standard generator of $H^1(S^1, \mathbb{Z})$.*

PROOF. Let θ be a locally inner automorphism of $C_0(T, \mathcal{K})$ with $\zeta(\theta) = [p]$ in $H^2(T, \mathbb{Z})$, and let $(A, \alpha) = \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(C_0(T, \mathcal{K}), \theta)$ as in §3(a). By Corollary 3.5, $A \rtimes_\alpha \mathbb{R}$ has spectrum the total space of the principal $\mathbb{T}$-bundle over T with characteristic class $\zeta(\theta)$, so $(A \rtimes_\alpha \mathbb{R})^\wedge \to T$ may be identified with $p\colon X \to T$. Furthermore, that corollary also gives

$$\delta(A) = z \times [p], \qquad \delta(A \rtimes_\alpha \mathbb{R}) = 0.$$

Since $A \rtimes_\alpha \mathbb{R}$ is stable, $A \rtimes_\alpha \mathbb{R} \cong C_0(X, \mathcal{K})$, and the dual action $\hat{\alpha}$ of $\mathbb{R}$ on $A \rtimes_\alpha \mathbb{R}$ induces the bundle map p. Thus by Corollary 4.4, $\hat{\alpha}$ is exterior equivalent to $\beta \otimes \mathrm{id}$ (acting on $C_0(X) \otimes \mathcal{K}$). It follows that we have strong Morita equivalences

$$C_0(X) \rtimes_\beta \mathbb{R} \sim (A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R} \sim A$$

(the last of these by Takai duality), and so $(C_0(X) \rtimes_\beta \mathbb{R})^\wedge \cong \hat{A} \cong \mathbb{T} \times T$ and $\delta(C_0(X) \rtimes_\beta \mathbb{R}) = \delta(A) = z \times [p]$, as asserted. □

EXAMPLE 4.6. For instance, suppose $X = S^3$, viewed as the unit sphere in $\mathbb{C}^2$, and β is given by $t \cdot (w, z) = (e^{2\pi i t} w, e^{2\pi i t} z)$. Then $p\colon X \to T$ is the Hopf fibration $S^3 \to S^2$, and $[p]$ is the standard generator of $H^2(S^2, \mathbb{Z})$. Thus $C(X) \rtimes_\beta \mathbb{R}$ is a continuous-trace algebra with spectrum $S^1 \times S^2$ and Dixmier-Douady class $z \times [p]$, which generates $H^3(S^1 \times S^2, \mathbb{Z}) \cong \mathbb{Z}$.

Actually, one could have predicted this by other means. By [**39**, Theorem 5.1], we knew $C(X) \rtimes_\beta \mathbb{R}$ has continuous trace, and by [**38**, Theorem 5.3], $(C(X) \rtimes_\beta \mathbb{R})^\wedge \cong S^1 \times S^2$. But by Connes' Thom isomorphism theorem [**3**], we must have

$$\begin{cases} K_0\big(C(X) \rtimes_\beta \mathbb{R}\big) \cong K_1(C(X)) = K^1(S^3) \cong \mathbb{Z}, \\ K_1\big(C(X) \rtimes_\beta \mathbb{R}\big) \cong K_0(C(X)) = K^0(S^3) \cong \mathbb{Z}. \end{cases}$$

Since $K^0(S^1 \times S^2) \cong K^1(S^1 \times S^2) \cong \mathbb{Z}^2$, this shows $\delta(C(X) \rtimes_\beta \mathbb{R})$ cannot be zero. In fact, if A is a continuous-trace algebra with spectrum $S^1 \times S^2$ and Dixmier-Douady class δ, one easily sees from [**30**, Theorem 6.5] that if $K_0(A) \cong K_1(A) \cong \mathbb{Z}$, then δ must be a generator of $H^3(S^1 \times S^2, \mathbb{Z})$. □

In order to extend Corollary 4.4 to the case of nontrivial continuous-trace algebras, we shall show below (Theorem 4.10 and Theorem 4.12) that it is often possible to lift an $\mathbb{R}$-action on a space X to an action on a *nontrivial* continuous-trace algebra A over X, in spite of the fact that the group extension

$$1 \to \mathrm{Aut}_{C_0(X)} A \to \mathrm{Aut}(A) \to \mathrm{Homeo}_\delta(X)$$

of Theorem 0.5(c) usually does not split. This once again is due to very special properties of the real line.

First, however, let us discuss the implications of our work with respect to a problem discussed in [**31**], namely that of describing the topology on $(A \rtimes_\alpha \mathbb{R})^\wedge$ or $\mathrm{Prim}(A \rtimes_\alpha \mathbb{R})$ when α is an action of $\mathbb{R}$ on a separable type I C^*-algebra A. This had been expected to be a tractable problem, at least when $\hat{A}/\mathbb{R}$ is T_0, because of the fact that $H^2(G, \mathbb{T}) = 0$ for every closed subgroup G of $\mathbb{R}$, so that "Mackey obstructions" never arise. Thus one of us has conjectured in [**31**] that results of D. Williams [**38**, Theorem 5.3] regarding the topology of $\mathrm{Prim}(C_0(\Omega) \rtimes_\alpha G)$, (G, Ω) any separable locally compact transformation group with G abelian, could be carried over to this setting. The precise conjecture was that if $\Omega = \hat{A}$ and one defines an equivalence relation on $\hat{\mathbb{R}} \times \Omega$ by

$$(4.7) \qquad (\gamma, x) \sim (\chi, y) \Leftrightarrow \overline{\mathbb{R} \cdot x} = \overline{\mathbb{R} \cdot y} \quad \text{and} \quad \gamma\bar{\chi} \in (\mathrm{Stab}_x)^\perp,$$

then $\mathrm{Prim}(A \rtimes_\alpha \mathbb{R})$ should be homeomorphic to $\hat{\mathbb{R}} \times \Omega/\sim$. (When $\hat{A}/\mathbb{R}$ is T_0, the crossed product will be type I and one can replace $\mathrm{Prim}(A \rtimes_\alpha \mathbb{R})$ by $(A \rtimes_\alpha \mathbb{R})^\wedge$ and delete the closure signs over $\mathbb{R} \cdot x$ and $\mathbb{R} \cdot y$ in (4.7).) The homeomorphism (say in the case $\hat{A}/\mathbb{R}$ is T_0) was to be given by the map φ: $\hat{\mathbb{R}} \times \Omega \to (A \rtimes_\alpha \mathbb{R})^\wedge$ defined by

$$\varphi(\gamma, x) = \mathrm{Ind}_{\mathrm{Stab}_x}^{\mathbb{R}}\left(\gamma|_{\mathrm{Stab}_x} \otimes \tilde{x}\right),$$

where $\tilde{x}$ is an extension of $x \in \hat{A}$ to the crossed product of A by the stabilizer of x. (The existence of $\tilde{x}$ comes from vanishing of the Mackey obstruction at x.)

Unfortunately, we see now that this conjecture is false, the problem being that even if the Mackey obstruction vanishes, there is generally no *canonical* way of extending x to $\tilde{x}$. For a specific counterexample, suppose $\Omega = \hat{A}$ is Hausdorff and every point of Ω has stabilizer $\mathbb{Z} \subset \mathbb{R}$. Then (4.7) simplifies greatly and $\hat{\mathbb{R}} \times \Omega/\sim$ is just $\hat{\mathbb{Z}} \times T$, where $T = \Omega/\mathbb{R}$. (The map $\Omega \to T$ is a locally trivial $\mathbb{T}$-bundle by [**6**].) However, specializing the remark following (3.2) to our setting shows that $(A \rtimes_\alpha \mathbb{R})^\wedge \to T$ can be arranged to yield *any* $\mathbb{T}$-bundle over T, not just the trivial bundle. Similarly, the analogue of this conjecture for actions of $\mathbb{Z}$ is false, since a noninner locally inner automorphism θ yields a nontrivial $\mathbb{T}$-bundle $(A_\theta \rtimes \mathbb{Z})^\wedge \to \hat{A}$.

If however, we require A to be a continuous-trace algebra with trivial Dixmier-Douady invariant, then by combining Corollary 4.4 with Williams' theorem, we can prove the conjecture of [**31**] after all.

THEOREM 4.8. *Let A be a separable continuous-trace algebra with spectrum* Ω, *and with* $\delta(A) = 0$ *in* $H^3(\Omega, \mathbb{Z})$, *and assume* $H^n(\Omega, \mathbb{Z})$ *is countable for* $n \leqslant 2$. *Let* α: $\mathbb{R} \to \mathrm{Aut}(A)$ *be any one-parameter automorphism group of A. Then* $\mathrm{Prim}(A \rtimes_\alpha \mathbb{R})$ (*or* $(A \rtimes_\alpha \mathbb{R})^\wedge$, *if* $\Omega/\mathbb{R}$ *is* T_0) *is homeomorphic to* $(\hat{\mathbb{R}} \times \Omega)/\sim$ *with the quotient topology, where* $\sim$ *is defined by* (4.7). *In fact, this homeomorphism is equivariant for the dual action of* $\hat{\mathbb{R}}$.

PROOF. Since $(A \otimes \mathcal{K}) \rtimes_{\alpha \otimes \mathrm{id}} \mathbb{R} \cong (A \rtimes_\alpha \mathbb{R}) \otimes \mathcal{K}$ (equivariantly for the dual action of $\hat{\mathbb{R}}$), it is no loss of generality to assume A is stable. Then A is isomorphic to $C_0(\Omega) \otimes \mathcal{K}$, and by Corollary 4.4, α is exterior equivalent to $\tau \otimes \mathrm{id}$, where τ is the action of $\mathbb{R}$ on $C_0(\Omega)$ associated to the action induced by α on $\hat{A}$. Thus (by Theorem 0.10) there is an $\hat{\mathbb{R}}$-equivariant isomorphism of $A \rtimes_\alpha \mathbb{R}$ with $(C_0(\Omega) \rtimes_\tau \mathbb{R}) \otimes \mathcal{K}$. Now apply [**38**, Theorem 5.3] to $C_0(\Omega) \rtimes_\tau \mathbb{R}$. □

REMARK 4.9. In spite of the fact that the groups $\mathbb{Z}$ and $\mathbb{T}$ also have the property that all their closed subgroups G satisfy $H^2(G, \mathbb{T}) = 0$, the analogue of Theorem 4.8 fails for these groups, as one can see by considering locally unitary actions and using Theorem 0.16.

Furthermore, the need for the hypothesis that $\delta(A) = 0$ in Theorem 4.6 may be attributed to the fact that only in this case is there a natural lifting of homeomorphisms of Ω to automorphisms of A. To see that

$$1 \to \mathrm{Aut}_{C_0(\Omega)} A \to \mathrm{Aut}(A) \to \mathrm{Homeo}_{\delta(A)}(\Omega)$$

usually does not split when $\delta(A) \neq 0$, note that if $H^3(\Omega, \mathbb{Z})$ is countable, any action of a connected locally compact abelian group G ($\mathbb{T}$, for example) on Ω must preserve $\delta(A)$. If the action were free and proper and lifted to an action α: $G \to \mathrm{Aut}(A)$, then by Theorem 1.1, $\delta(A)$ would have to be the pull-back of a class in $H^3(\Omega/G, \mathbb{Z})$, which is often not the case. (For instance, if $\Omega = S^3$ and G is $\mathbb{T}$ acting by the Hopf map, then 0 is the only element of $H^3(\Omega, \mathbb{Z})$ which is a pull-back from $H^3(S^2, \mathbb{Z})$, hence the free $\mathbb{T}$-action on Ω can be lifted to an action on A only if $\delta(A) = 0$.) □

We come now to an existence theorem for $\mathbb{R}$-actions on nontrivial continuous-trace algebras. A special case was proved in Example 3.7.

THEOREM 4.10. *Let M be a smooth manifold, not necessarily compact, and let A be any stable continuous-trace algebra with spectrum M. Let* φ: $\mathbb{R} \to \mathrm{Diff}(M)$ *be any smooth action of* $\mathbb{R}$ *on M. Then there is a continuous action* α: $\mathbb{R} \to \mathrm{Aut}(A)$ *inducing* φ *on* $M = \hat{A}$. (*By Corollary* 4.3, *this action is unique up to exterior equivalence if* $H^n(M, \mathbb{Z})$ *is countable for* $n \leqslant 2$.)

PROOF. Without loss of generality, we may assume M connected (otherwise work one component at a time). Let $\delta = \delta(A) \in H^3(M, \mathbb{Z})$. Then δ is represented by a homotopy class of maps $M \to K(\mathbb{Z}, 3)$. It will be convenient to note that if $\mathcal{G} = \mathcal{U}/\mathbb{T}$, where $\mathcal{U}$ is the unitary group of an infinite-dimensional separable Hilbert space with the *norm* topology, then $\mathcal{G}$ is a Banach Lie group with Lie algebra

$$\mathfrak{g} = \{\text{selfadjoint bounded operators}\}/\{\text{scalar operators}\}.$$

Furthermore, since $\mathscr{U}$ is contractible by [**18**], $\mathscr{G}$ is a $K(\mathbb{Z},2)$, and δ may be viewed as the classifying map for a principal $\mathscr{G}$-bundle over M. In fact, from the exact sequences

$$1 \to \underset{\sim}{\mathbb{T}} \to \underset{\sim}{\mathscr{U}} \to \underset{\sim}{\mathscr{G}} \to 1,$$

$$0 \to \mathbb{Z} \to \underset{\sim}{\mathbb{R}} \to \underset{\sim}{\mathbb{T}} \to 0$$

of sheaves of germs of C^∞ functions over M, we have $H^1(M,\underset{\sim}{\mathscr{G}}) \cong H^2(M,\underset{\sim}{\mathbb{T}}) \cong H^3(M,\mathbb{Z})$, so that δ corresponds to a class in $H^1(M,\underset{\sim}{\mathscr{G}})$, hence to a *smooth* principal $\mathscr{G}$-bundle p: $E \to M$. We claim it is enough to lift φ to a smooth one-parameter group $\tilde{\varphi}$ of $\mathscr{G}$-bundle automorphisms of E. For indeed, we may realize A as $\Gamma_0(M,\mathscr{E})$, where $\mathscr{E} = E \times_\rho \mathscr{K}$ is the bundle of elementary C^*-algebras associated to E via the standard map ρ: $\mathscr{G} \to \operatorname{Aut}(\mathscr{K})$, and then α may be constructed out of $\tilde{\varphi}$ in the obvious fashion (since by assumption, $\tilde{\varphi}_t$ on E commutes with the $\mathscr{G}$-action on E and so passes to $\mathscr{E}$).

Hence we are reduced to the problem of lifting φ to the principal bundle E. If $\mathscr{G}$ were a (finite-dimensional) Lie group, it would be standard (see, for instance, [**17**, Chapter 2]) that one can do this by choosing a connection on the bundle, lifting the vector-field $\dot{\varphi}$ that generates φ to a horizontal vector field Y on E, and then taking the integral curves of Y. The same method works here except for the minor complications due to the fact that $\mathscr{G}$ is infinite dimensional. Here is a brief summary of the main points. At any point $e \in E$ with $p(e) = m \in M$, the vectors tangent to the $\mathscr{G}$-orbit through e define the *vertical tangent space* $V_e \subset T_eE$. Since E is locally diffeomorphic to $M \times \mathscr{G}$, $V_e \cong \mathfrak{g}$ is of finite codimension in T_eE, and one can choose a *horizontal space* H_e complementary to it so that dp: $H_e \to T_mM$ is an isomorphism. Once H_e is chosen at one point $e \in p^{-1}(m)$, there is a unique $\mathscr{G}$-equivariant choice of horizontal spaces at the other points in the same fiber. Thus one can choose a *connection*, i.e., a smoothly varying $\mathscr{G}$-equivariant choice of horizontal spaces in E. This is done as follows: a trivialization of E over some open set U in M gives a local product structure $p^{-1}(U) \cong U \times \mathscr{G}$ and thus an obvious connection over U. Then connections can be patched globally using a partition of unity, due to the fact that the connections over U form an affine space modeled on $T^*U \otimes \mathfrak{g}$, the 1-forms with values in $\mathfrak{g}$. Once one has a connection on E, φ may be lifted by integrating the linear first-order differential equation

$$\begin{cases} \dfrac{d}{dt}\tilde{\varphi}_t(e) = \text{horizontal lift of } \dfrac{d}{dt}\varphi_t(p(e)), \\ \tilde{\varphi}_0(e) = e. \quad \square \end{cases}$$

We do not know whether or not Theorem 4.10 still holds if M is replaced by an arbitrary space Ω and φ by any continuous flow on Ω. However, the theorem is more powerful than it looks, since it gives liftings in any situation where both A and the $\mathbb{R}$-action on $\hat{A}$ are pulled back from a smooth action on a manifold. This is the basis of our next main result, Theorem 4.12. However, we pause first to mention another application of Theorem 4.10.

COROLLARY 4.11. *Let M be any smooth manifold equipped with a free, minimal (i.e., every orbit is dense), smooth action of $\mathbb{R}$. Then for each $\delta \in H^3(M, \mathbb{Z})$, there is an associated simple stable (nuclear) C^*-algebra A_δ. If $H^n(M, \mathbb{Z})$ is countable for $n \leqslant 2$, then A_δ is uniquely determined. In general, the A_δ's corresponding to different values of δ will be nonisomorphic.*

PROOF. Let C_δ be the stable continuous-trace algebra with spectrum M and Dixmier-Douady class δ, and use Theorem 4.10 to construct an action α_δ of $\mathbb{R}$ on C_δ inducing the given flow on M. Of course α_δ is not unique, but under the countability hypothesis of Corollary 4.3, it is unique up to exterior equivalence. Let $A_\delta = C_\delta \rtimes_{\alpha_\delta} \mathbb{R}$, stabilized if necessary. This algebra is simple by the Gootman-Rosenberg proof of the generalized Effros-Hahn conjecture [9], and only depends on the exterior equivalence class of α_δ. By [3], $K_0(A_\delta) \cong K_1(C_\delta)$ and $K_1(A_\delta) \cong K_0(C_\delta)$, and the K-theory of C_δ usually varies as δ varies (as partially explained by [30, Theorem 6.5]).

To obtain a specific example of Corollary 4.11, one may take $M = T^3$ with a Kronecker flow on it. In this case, $K_*(A_\delta)$ varies with $|\delta|$. □

THEOREM 4.12. *Let T be any second-countable locally compact space with the homotopy type of a finite CW-complex, and $p\colon \Omega \to T$ any principal $\mathbb{T}$-bundle over T, A a stable continuous-trace algebra with spectrum Ω. Then there is an action α of $\mathbb{R}$ on A, unique up to exterior equivalence, such that every point in $\Omega = \hat{A}$ has stabilizer $\mathbb{Z}$ and the $\mathbb{R}$-action on Ω factors through the $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$-action defining p.*

Furthermore, $(A \rtimes_\alpha \mathbb{R})^\wedge$ together with the dual action of $\mathbb{R}$ defines another principal $\mathbb{T}$-bundle $\pi\colon (A \rtimes_\alpha \mathbb{R})^\wedge \to T$, and the characteristic classes $[p]$ and $[\pi]$ of the bundles p and π are related to the Dixmier-Douady classes by the equations

$$[\pi] = p_!\delta(A), \qquad [p] = \pi_!\delta(A \rtimes_\alpha \mathbb{R}),$$

where $p_!\colon H^3(\Omega, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ and $\pi_!\colon H^3((A \rtimes_\alpha \mathbb{R})^\wedge, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ are the Gysin maps of [36, *Theorem* VIII, 5.12]. *The class $\delta(A \rtimes_\alpha \mathbb{R})$ may be computed from $[p]$ and $\delta(A)$ in a manner to be specified below.*

PROOF. First we will show that $p\colon \Omega \to T$ is pulled back from a smooth $\mathbb{T}$-bundle over a manifold, in such a way that $\delta(A)$ is pulled back from a class in H^3 of the total space. This will prove existence of α because of Theorem 4.10. Then uniqueness will follow from Corollary 4.3. The formulas for $[\pi]$ and $\delta(A \rtimes_\alpha \mathbb{R})$ will be obtained by deriving them first in the case of "universal examples" and then pulling back.

Let X be the homotopy fiber of the map $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 4)$ corresponding to the cup product. The space X will have the universal property that given a space W with the homotopy type of a CW-complex, and given classes a, $b \in H^2(W, \mathbb{Z})$ such that $a \cup b = 0$ in $H^4(W, \mathbb{Z})$, the pair (a, b) is pulled back from a map $W \to X$. (More precisely, there is an $f\colon W \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$, unique up to homotopy, such that $a = f^*(\iota_1)$, $b = f^*(\iota_2)$, where ι_1 and ι_2 are the canonical generators of $H^2(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$, and f may be lifted to a map $W \to X$.) An easy calculation with the homotopy sequence and Serre spectral sequence of the fibration

$$\begin{array}{ccc} X & \to & K(\mathbb{Z},2)\times K(\mathbb{Z},2) \\ & & \downarrow \\ & & K(\mathbb{Z},4) \end{array}$$

shows that X is simply connected, and that

$$H^2(X,\mathbb{Z})\cong \mathbb{Z}\beta_1\oplus \mathbb{Z}\beta_2,\quad H^3(X,\mathbb{Z})=0,\quad H^4(X,\mathbb{Z})\cong \mathbb{Z}\beta_1^2\oplus \mathbb{Z}\beta_2^2.$$

Here β_1 and β_2 are the images of ι_1 and ι_2, and by construction, $\beta_1\cup\beta_2=0$. Then let $q\colon E\to X$ be the principal $\mathbb{T}$-bundle with $[q]=\beta_1$. The Gysin sequence

$$0=H^3(X,\mathbb{Z})\overset{q^*}{\to}H^3(E,\mathbb{Z})\overset{q_!}{\to}H^2(X,\mathbb{Z})\overset{[q]\cup}{\to}H^4(X,\mathbb{Z})$$

shows that $H^3(E,\mathbb{Z})$ is infinite cyclic, with a unique generator σ such that $q_!(\sigma)=\beta_2$.

Now let us return to our given $p\colon \Omega\to T$. From the Gysin sequence

$$H^3(T,\mathbb{Z})\overset{p^*}{\to}H^3(\Omega,\mathbb{Z})\overset{p_!}{\to}H^2(T,\mathbb{Z})\overset{[p]\cup}{\to}H^4(T,\mathbb{Z}),$$

we have $[p]\cup p_!(\delta(A))=0$. Thus there is a map $f\colon T\to X$ such that $([p], p_!\delta(A))=f^*(\beta_1,\beta_2)$. Since $[p]=f^*\beta_1$, $p=f^*q$, and $p_!\bar f^*(\sigma)=f^*q_!(\sigma)=f^*\beta_2=p_!\delta(A)$, where $\bar f\colon \Omega\to E$ completes the diagram

$$\begin{array}{ccc} \Omega & \overset{\bar f}{\to} & E \\ \downarrow p & & \downarrow q \\ T & \overset{f}{\to} & X \end{array}$$

Rewriting this, we have $p_!(\delta(A)-\bar f^*(\sigma))=0$, hence (by the Gysin sequence) $\delta(A)-\bar f^*(\sigma)\in p^*(H^3(T,\mathbb{Z}))$. In other words, there is a map $g\colon T\to K(\mathbb{Z},3)$ such that if we form the pull-back diagram

$$\begin{array}{ccc} \Omega & \overset{\bar f\times(g\circ p)}{\to} & E\times K(\mathbb{Z},3) \\ \downarrow p & & \downarrow q\times \mathrm{id} \\ T & \overset{f\times g}{\to} & X\times K(\mathbb{Z},3), \end{array}$$

then (with ι_3 the standard generator of $H^3(K(\mathbb{Z},3),\mathbb{Z})$)

$$[p]=(f\times g)^*(\beta_1,0),\quad \delta(A)=(\bar f\times g)^*((\sigma,0)+(0,\iota_3)).$$

This shows that all the given data is pulled back from the "universal example" $q\times\mathrm{id}\colon E\times K(\mathbb{Z},3)\to X\times K(\mathbb{Z},3)$. Now if T has the homotopy type of a finite complex, there is a finite subcomplex Y of $X\times K(\mathbb{Z},3)$ (which can be chosen to have the same cohomology through degree 4) such that $f\times g$ factors (up to homotopy) through Y. We may "fatten" Y to an open manifold $\overline{M}$, by embedding Y in some Euclidean space and taking a regular neighborhood. Let M be the total space of the $\mathbb{T}$ bundle over $\overline{M}$ coming from $q \times \mathrm{id}$. Then $M \to \overline{M}$ may be chosen to be a smooth bundle, so that $\overline{M}$ is the quotient of M by a smooth free action of $\mathbb{T}$. By construction, A is the pull-back of a continuous-trace algebra over M. So applying Theorem 4.10, we get an $\mathbb{R}$-action α on A inducing the bundle p. It is essentially unique by Corollary 4.3.

Having established existence and essential uniqueness of the action α, we proceed to compute $A \rtimes_\alpha \mathbb{R}$. By Theorem 2.2, the restriction of representations to A defines another principal $\mathbb{T}$-bundle $\pi\colon (A \rtimes_\alpha \mathbb{R})^\wedge \to T$ (since $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong \hat{\mathbb{Z}}$) and we have the commutative diagram of principal $\mathbb{T}$-bundles

$$\begin{array}{ccccc} & & (A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z})^\wedge & & \\ & \pi^*p \swarrow & & \searrow p^*\pi & \\ (A \rtimes_\alpha \mathbb{R})^\wedge & & & & \hat{A} \\ & \pi \searrow & & \swarrow p & \\ & & \hat{A}/\mathbb{R} & & \end{array}$$

We begin by establishing the identity $[\pi] = p_!\delta(A)$. This is done by considering two cases.

Case 1. $[p] = 0$ in $H^2(T, \mathbb{Z})$. To handle this case, note that if p is trivial, so $\Omega \cong \mathbb{T} \times T$, then $H^3(\Omega, \mathbb{Z}) \cong p^*H^3(T, \mathbb{Z}) \oplus z \times H^2(T, \mathbb{Z})$, where z is the standard generator of $H^1(\mathbb{T}, \mathbb{Z})$. By definition, $p_! \equiv 0$ on $p^*H^3(T, \mathbb{Z})$, while $p_!(z \times x) = x$ if $x \in H^2(T, \mathbb{Z})$. Now given any continuous-trace algebra A with spectrum Ω, we have a unique decomposition of $\delta(A)$ as

$$\delta(A) = p^*(y) + z \times p_!(\delta(A)), \qquad y \in H^3(T, \mathbb{Z}).$$

Let D be the stable continuous-trace algebra with spectrum T and Dixmier-Douady class y, and using Theorem 0.5, let θ be a locally inner automorphism of D associated to $p_!(\delta(A))$. By Corollary 3.5, together with the uniqueness result Corollary 4.3, $(A, \alpha) \cong \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(D, \theta)$, up to exterior equivalence of the actions, and if $\pi\colon (A \rtimes_\alpha \mathbb{R})^\wedge \to T$ is the restriction map, then $[\pi] = p_!(\delta(A))$. The same calculation also shows that $\delta(A \rtimes_\alpha \mathbb{R}) = \pi^*\delta(D) = \pi^*(y)$.

Case 2. The general case. By the existence part of the proof, (A, α) was pulled back from M (notation as above), so we may assume $\Omega = M$, $T = \overline{M}$, where M and $\overline{M}$ are simply connected manifolds with torsion-free cohomology through degree 4, coinciding with that of $E \times K(\mathbb{Z}, 3)$ and $X \times K(\mathbb{Z}, 3)$, respectively. Recall that by construction, $H^2(\overline{M}, \mathbb{Z}) \cong \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$, $H^3(\overline{M}, \mathbb{Z}) \cong \mathbb{Z}\iota_3$, $H^4(\overline{M}, \mathbb{Z}) \cong \mathbb{Z}\beta_1^2 \oplus \mathbb{Z}\beta_2^2$, and $[p] = \beta_1$. From the Gysin sequence for p, $H^3(M, \mathbb{Z}) \cong \mathbb{Z}p^*\iota_3 \oplus \mathbb{Z}\sigma$, where $p_!\sigma = \beta_2$, and $H^2(M, \mathbb{Z}) \cong \mathbb{Z}p^*\beta_2$. We are supposing that $\delta(A) = p^*\iota_3 + \sigma$, so that $p_!\delta(A) = \beta_2$.

CLAIM. $[\pi] = \beta_2$, and $\delta(A \rtimes_\alpha \mathbb{R}) = \pi^*\iota_3 + \rho$, where $\pi_!\rho = \beta_1$. In other words, $(A \rtimes_\alpha \mathbb{R}, \hat{\alpha})$ is obtained from (A, α) by reversing the roles of β_1 and β_2 and keeping all else the same.

PROOF. Let $[\pi] = n\beta_2 + m\beta_1$, so that $[p^*\pi] = np^*\beta_2$. First of all, we observe that $n = 1$. For this we use Case 1. Since (by construction) $\pi_2(\overline{M})$ is free abelian of rank 2 and we may take $\dim \overline{M} \geqslant 5$, the homology class dual to β_2 may be represented by a smoothly embedded 2-sphere S. (By Hurewicz, this homology class is represented by a map $S^2 \to \overline{M}$, which can be approximated by an embedding by [**15**, Theorem 2.13].) Since $[p] = \beta_1$, $p \mid p^{-1}(S)$ is trivial and $p^{-1}(S) \cong \mathbb{T} \times S$. Restricting α to $A|_{p^{-1}(S)}$, we may apply Case 1 to see that the characteristic class of

$$\pi_S\colon \left(A|_{p^{-1}(S)} \rtimes \mathbb{R}\right)^\wedge \to S$$

is $i^*(\beta_2)$, where $i\colon S \to \overline{M}$ is the inclusion. Since $\pi_S = \pi \mid \pi^{-1}(S)$, $[\pi_S] = i^*[\pi]$, which establishes our claim.

Next we show that $m = 0$. For this let $Q = (A \rtimes_\alpha \mathbb{R})^\wedge$, $P = (A \rtimes_{\alpha|_{\mathbf{Z}}} \mathbb{Z})^\wedge$. By Corollary 2.5, $(p^*\pi)^*\delta(A) = (\pi^*p)^*\delta(A \rtimes_\alpha \mathbb{R})$. From the exact Gysin sequences

$$\to H^2(\overline{M}, \mathbb{Z}) \overset{\beta_1 \cup -}{\to} H^4(\overline{M}, \mathbb{Z}) \overset{p^*}{\to} H^4(M, \mathbb{Z}) \overset{p_!}{\to},$$

$$0 = H^1(M, \mathbb{Z}) \to H^3(M, \mathbb{Z}) \overset{(p^*\pi)^*}{\to} H^3(P, \mathbb{Z}) \overset{(p^*\pi)_!}{\to} H^2(M, \mathbb{Z}) \overset{p^*\beta_2 \cup -}{\to} H^4(M, \mathbb{Z}),$$

we see that $p^*\beta_2^2$ generates an infinite cyclic summand in $H^4(M, \mathbb{Z})$, so that $(p^*\pi)_!\colon H^3(P, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is zero and $(p^*\pi)^*\colon H^3(M, \mathbb{Z}) \to H^3(P, \mathbb{Z})$ is an isomorphism.

Now if $m \neq 0$, the cup product with $[\pi] = \beta_2 + m\beta_1$ is injective from $H^2(\overline{M}, \mathbb{Z})$ to $H^4(\overline{M}, \mathbb{Z})$, so from the exact Gysin sequence

$$0 \to H^3(\overline{M}, \mathbb{Z}) \overset{\pi^*}{\to} H^3(Q, \mathbb{Z}) \overset{\pi_!}{\to} H^2(\overline{M}, \mathbb{Z}) \overset{[\pi] \cup -}{\to} H^4(\overline{M}, \mathbb{Z}),$$

we see that $H^3(Q, \mathbb{Z}) = \pi^* H^3(\overline{M}, \mathbb{Z})$. This is impossible, since then the Dixmier-Douady class of $A \rtimes_\alpha \mathbb{R}$ would have to be pulled back from $\overline{M}$. This would give $A \rtimes_\alpha \mathbb{R} \cong \pi^*D$, D some continuous-trace algebra with spectrum $\overline{M}$. By Corollary 4.3 again, this would imply $\hat{\alpha}$ was exterior equivalent to $\pi^*\mathrm{id}$ (lifted to an action of $\mathbb{R}$), and (viewing A by duality as being essentially $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}$) we would have to have $[p] = 0$, which is not the case. Thus, $m = 0$, i.e., $[\pi] = \beta_2$.

Now by the Gysin sequence for π when $[\pi] = \beta_2$, there is a unique class $\rho \in H^3(Q, \mathbb{Z})$ with $\pi_!(\rho) = \beta_1 = [p]$. By the relation that

$$(p^*\pi)^*\delta(A) = (\pi^*p)^*\delta(A \rtimes_\alpha \mathbb{R}) \quad \text{in } H^3(P, \mathbb{Z}),$$

we obtain $\delta(A \rtimes_\alpha \mathbb{R}) = \pi^*\iota_3 + \rho$. As required by symmetry, $\pi_!(A \rtimes_\alpha \mathbb{R}) = \pi_!(p) = \beta = [p]$. This completes the proof in the case $\hat{A} = M$, and by naturality of the Gysin maps, the relations

$$[\pi] = p_!\delta(A), \quad [p] = \pi_!\delta(A \rtimes_\alpha \mathbb{R})$$

hold also in general. To compute $\delta(A \rtimes_\alpha \mathbb{R})$ in the general case, we repeat the recipe implicit above: choose $f\colon T \to X$ such that $[p] = f^*\beta_1$, $p_!\delta(A) = f^*\beta_2$, and choose $g\colon T \to K(\mathbb{Z}, 3)$ such that $\delta(A) = \bar{f}^*\sigma + (g \circ p)^*\iota_3$. Then $\delta(A \rtimes_\alpha \mathbb{R}) - (g \circ \pi)^*\iota_3$ is the class in $H^3(Q, \mathbb{Z})$, where $\pi\colon Q \to T$ has characteristic class $[\pi] = p_!\delta(A)$, which is pulled back from ρ. □

EXAMPLE 4.13. Using Theorem 4.12, we may complete the calculations of what happens in Examples 3.6 and 3.7. In Example 3.6, $\Omega = \mathbb{RP}^5 \times \mathbb{RP}^5$, $T = \mathbb{CP}^2 \times \mathbb{RP}^5$, and $\pi\colon (A \rtimes_\alpha \mathbb{R})^\wedge \to T$ is the bundle $\mathbb{CP}^2 \times E \to \mathbb{CP}^2 \times \mathbb{RP}^5$ (there called

q). If x is the standard generator of $H^2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$ and y is the generator of $H^2(\mathbb{RP}^5, \mathbb{Z}) \cong \mathbb{Z}_2$, then $[p] = 2x \times 1$, $[\pi] = 1 \times y$.

Furthermore, $\delta(A)$ is the nontrivial element of $H^3(\Omega, \mathbb{Z}) \cong \mathbb{Z}_2$. As required, $p_!\delta(A) = [\pi]$, because

$$(1 \times y) \cup [p] = 2x \times y = x \times 2y = 0,$$

and so $1 \times y$ is in the image of $p_!$. To compute $\delta(A \rtimes_\alpha \mathbb{R}) = \eta$, note that $\pi_!(\eta) = [p] = 2x \times 1$. Now a calculation with the Künneth Theorem shows $H^3(T, \mathbb{Z}) = 0$, hence $\pi_!$: $H^3(E, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ is injective and this determines η uniquely. (In fact, $H^1(E, \mathbb{Z})$ is infinite cyclic with some generator w, and $\eta = x \times w$ generates $H^3(E, \mathbb{Z})$.) Finally, $\delta(A \rtimes_\alpha \mathbb{Z})$ is the common pull-back of $\delta(A)$ and of $\delta(A \rtimes_\alpha \mathbb{R})$ to $H^3(\mathbb{RP}^5 \times E)$, which is $y \times w$.

In Example 3.7, $\Omega = S^3$, $T = S^2$, $[p] = x$, the standard generator of $H^2(S^2, \mathbb{Z})$, and δ can be any multiple nu of the standard generator u of $H^3(S^3, \mathbb{Z})$. Since $p_!(u) = x$, $[\pi] = p_!(nu) = nx$, and $(A \rtimes_\alpha \mathbb{R})^\wedge$ is the total space of the circle bundle π: $X \to S^2$. Since $H^3(T, \mathbb{Z}) = 0$, $\delta(A \rtimes_\alpha \mathbb{R}) = \eta$ is uniquely determined by the condition $\pi_!(\eta) = [p] = x$. Since $H^2(\Omega, \mathbb{Z}) = 0$, the bundle $(A \rtimes_\alpha \mathbb{Z})^\wedge \to \Omega$ is trivial, so $(A \rtimes_\alpha \mathbb{Z})^\wedge \cong S^1 \times S^3$, and $\delta(A \rtimes_\alpha \mathbb{Z})$ is just $1 \times \delta$. □

To show how the above fits into a broader context, let us say a few words about computation of $A \rtimes_\alpha \mathbb{R}$, where A is a separable continuous-trace algebra and α: $\mathbb{R} \to \mathrm{Aut}(A)$ is an essentially arbitrary action with $\hat{A}/\mathbb{R}$ countably separated. Then we discuss some related problems in the structure theory of C^*-algebras of Lie groups. Of course the problem of computing $A \rtimes_\alpha \mathbb{R}$ will be unmanageable in general, but the following is true.

THEOREM 4.14. *Let A be a separable continuous-trace algebra, α an action of $\mathbb{R}$ on A such that $\hat{A}/\mathbb{R}$ is T_0. Then there exists an $\mathbb{R}$-invariant essential ideal $B = B_1 \oplus B_2 \oplus B_3$ in A, such that B_1, B_2, and B_3 are $\mathbb{R}$-invariant, and*

(1) $\mathbb{R}$ *acts freely and properly on* $\hat{B}_1$,

(2) $\mathbb{R}$ *acts trivially on* $\hat{B}_2$,

(3) *each point $x \in \hat{B}_3$ has cyclic stability group $c(x)\mathbb{Z}$, where c: $\hat{B}_3 \to (0, \infty)$ is a continuous function.*

Furthermore, for any B_j satisfying condition (j) $(j = 1, 2, 3)$, $B_j \rtimes_\alpha \mathbb{R}$ has continuous trace.

PROOF. By [7, Theorem 3], $\hat{A}$ contains an $\mathbb{R}$-invariant dense open subset on which stability groups are continuous. This open set will correspond to an $\mathbb{R}$-invariant essential ideal of A, so without loss of generality we may assume stability groups are continuous.

LEMMA 4.15. *Suppose Ω is a second-countable locally compact space equipped with an action of $\mathbb{R}$ for which $\Omega/\mathbb{R}$ is T_0. Define an $\mathbb{R}$-invariant function c: $\Omega \to [0, \infty]$ by*

$$c(x) = \begin{cases} 0 & \text{if } \mathbb{R} \cdot x = x, \\ \infty & \text{if } \mathbb{R} \text{ acts freely on the orbit } \mathbb{R} \cdot x, \\ \text{that number} > 0 \text{ such that } \mathbb{R}_x = c(x)\mathbb{Z}, & \\ \quad \text{if the action of } \mathbb{R} \text{ on } \mathbb{R} \cdot x \text{ is periodic.} & \end{cases}$$

Then continuity of the stability groups in the sense of [7] *is equivalent to continuity of the function c.*

PROOF. If c is continuous, it is easy to see that stability groups are continuous. Conversely, if one has continuity of the stability groups in the sense of Glimm, then given $x_n \to x$ and $s \in \mathbf{R}_x$, there exist $s_n \in \mathbf{R}_{x_n}$ with $s_n \to s$. If $c(x) = \infty$, this says nothing, but on the other hand $c(x_n)$ must tend to ∞ since otherwise one could pass to a subsequence and assume $c(x_n) \to c < \infty$, and one could choose $s_n \in \mathbf{R}_{x_n}$ with $s_n \to s \neq 0$, which would force $s \cdot x = x$, a contradiction. If $0 < c(x) < \infty$, the condition of Glimm guarantees that one can choose s_n divisible by $c(x_n)$ with $s_n \to c(x)$, so $\liminf c(x_n) \leqslant c(x)$, but again if $c(x_n) \nrightarrow c(x)$, $\mathbf{R}_x$ would properly contain $c(x)\mathbf{Z}$. Finally, if $c(x) = 0$, the Glimm condition forces $\liminf c(x_n) = 0$, and if $c(x_n) \nrightarrow 0$, one can pass to a subsequence on which $c(x_n) \to c > 0$ and obtain a contradiction. □

PROOF OF 4.14 (CONT'D). Assuming the stability groups are continuous, c is a continuous function by the lemma. Thus we may choose B_3 so that $\hat{B}_3 = \hat{A} - c^{-1}(\{0, \infty\})$, B_2 so that $\hat{B}_2 = \operatorname{int} c^{-1}(0)$. If we let $\hat{B}_1 = \operatorname{int} c^{-1}(\infty)$, then in general the action of $\mathbf{R}$ on $\hat{B}_1$ will not be proper; however, we can remedy this by passing to some dense open set, by [**10**, p. 95]. Then $B_1 \rtimes_\alpha \mathbf{R}$ has continuous trace by Theorem 1.1. Since $\mathbf{R}$ acts trivially on $\hat{B}_2$, the action of $\mathbf{R}$ on B_2 will be unitary by Corollary 0.14 and Remark 0.15, so that $B_2 \rtimes_\alpha \mathbf{R} \cong B_2 \otimes C_0(\mathbf{R})$, which obviously has continuous trace. Finally, the action of $\mathbf{R}$ on $\hat{B}_3$ has all isotropy groups isomorphic to $\mathbf{Z}$ and varying continuously. If the isotropy were actually constant, we could deduce from Theorem 2.2 that $B_3 \rtimes_\alpha \mathbf{R}$ has continuous trace, and assuming the spaces involved are nice enough we would even determine the crossed product explicitly by Theorem 4.12. Thus the proof of the theorem is completed by the following lemma. □

LEMMA 4.16. *Let A be a C^*-algebra with Hausdorff spectrum Ω, and suppose $A = \Gamma_0(E)$, where E is a continuous field of elementary C^*-algebras over Ω. Let α: $\mathbf{R} \to \operatorname{Aut}(A)$ be a continuous action such that for $x \in \Omega$, the stabilizer $\mathbf{R}_x$ of x in $\mathbf{R}$ is $c(x)\mathbf{Z}$, where c: $\Omega \to (0, \infty)$ is continuous. Define a new action β: $\mathbf{R} \to \operatorname{Aut}(A)$ (having all stability groups of points in Ω equal to $\mathbf{Z}$) by*

$$(\beta_s f)(x) = (\alpha_{c(x)s} f)(x), \quad \text{for } s \in \mathbf{R},\ x \in X,\ f \in \Gamma_0(E).$$

Then $A \rtimes_\alpha \mathbf{R} \cong A \rtimes_\beta \mathbf{R}$.

PROOF. We think of $A \rtimes_\alpha \mathbf{R}$ and $A \rtimes_\beta \mathbf{R}$ as completions of $C_c(\mathbf{R}, \Gamma_0(E))$, where this space has two different convolution multiplications, $\star$ and $\tilde{\star}$, and two different involutions, $*$ and $\tilde{*}$, respectively. Define

$$\Phi\colon C_c(\mathbf{R}, \Gamma_0(E)) \to C_c(\mathbf{R}, \Gamma_0(E)) \quad \text{by} \quad (\Phi h)(t)(x) = c(x)h(c(x)t)(x).$$

Then Φ is a $*$-algebra homomorphism from the $(\star, *)$-structure to the $(\tilde{\star}, \tilde{*})$-structure, since (viewing α and β as acting on both $\Gamma_0(E)$ and E)

$$\begin{aligned}(\Phi h)^{\tilde{*}}(t)(x) &= \beta_t\big((\Phi h)(-t)^*\big)(x)\\ &= \alpha_{c(x)t}\big((\Phi h)(-t)^*\big)(x) = \alpha_{c(x)t}\big(c(x)h(-c(x)t)^*(x)\big)\end{aligned}$$

(using the fact that c is constant on $\mathbb{R}$-orbits in X) and

$$\begin{aligned}\Phi(h^*)(t,x) &= c(x)h^*(c(x)t)(x)\\ &= c(x)\alpha_{c(x)t}\big(h(-c(x)t)^*\big)(x) = (\Phi h)^{\tilde{*}}(t)(x),\end{aligned}$$

and also

$$\begin{aligned}((\Phi h_1)\bigstar(\Phi h_2))(t)(x) &= \int [(\Phi h_1)(s)\beta_s((\Phi h_2)(t-s))](x)\,ds\\ &= \int (\Phi h_1)(s)(x)\alpha_{c(x)s}((\Phi h_2)(t-s))(x)\,ds\\ &= \int c(x)h_1(c(x)s)(x)\alpha_{c(x)s}(c(x)h_2(c(x)t-c(x)s))(x)\,ds\\ &= \int c(x)h_1(r)(x)\alpha_r(h_2(c(x)t-r))(x)\,dr \quad (\text{where } r=c(x)s)\\ &= c(x)(h_1\bigstar h_2)(c(x)t)(x) = \Phi(h_1\bigstar h_2)(t)(x).\end{aligned}$$

Since Φ is clearly invertible (with $(\Phi^{-1}h)(t)(x) = c(x)^{-1}h(c(x)^{-1}t)(x)$) and isometric for the L^1-norm on $L^1(\mathbb{R}, \Gamma_0(E))$, it extends to an isomorphism of the C^*-completions $A \rtimes_\alpha \mathbb{R}$ and $A \rtimes_\beta \mathbb{R}$. $\square$

We conclude this section with a few remarks about the C^*-algebras of Lie groups, especially of solvable Lie groups. The main motivation for the discussion of dual topologies in **[31]** came from the old problem of trying to determine if, for G a connected, simply connected exponential Lie group, the Kirillov-Bernat bijection

$$\mathfrak{g}^*/G \to \hat{G}$$

(here $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is its dual, on which G acts by the coadjoint action) is always a homeomorphism. For N a normal connected subgroup of codimension 1 in such a G, $C^*(G) \cong C^*(N) \rtimes_\alpha \mathbb{R}$, hence one could hope to solve this problem if one had good control over the dual topology for crossed products by $\mathbb{R}$.

Now as we have seen, the general conjecture in **[31]** about the topology of $(A \rtimes_\alpha \mathbb{R})^\wedge$ fails even when A has continuous trace, unless $\delta(A) = 0$, in which case Theorem 4.8 applies. Furthermore, algebras A with $\delta(A) \neq 0$ can arise themselves as crossed products $B \rtimes \mathbb{R}$ with B stably commutative (see Example 4.6), provided $\mathbb{R}$ has (at least some) periodic orbits on $\hat{B}$. So we are naturally led to the following problem and conjecture.

PROBLEM 4.17. If G is a Lie group and A is a continuous-trace subquotient of $C^*(G)$, when is $\delta(A) \neq 0$ possible?

CONJECTURE 4.18. If G is an exponential solvable Lie group and A is a continuous-trace subquotient of $C^*(G)$, then necessarily $\delta(A) = 0$.

The answer to Problem 4.17 is definitely "sometimes", as will be clear from the following two examples. Thus even if Conjecture 4.18 is correct, one probably could not prove a similar result for any larger nice class of Lie groups, except perhaps for those where the semisimple part acts trivially on the radical.

EXAMPLE 4.19. We shall construct a (disconnected) solvable Lie group G with abelian identity component, whose group C^*-algebra has continuous-trace quotient with nonzero Dixmier-Douady invariant. Let Q be the quaternion group of order 8, and let $G = \mathbb{C}^4 \rtimes Q$, where Q acts on $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ by the linear action in which the center $Z(Q)$ of Q acts trivially and the quotient $Q/Z(Q) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts by sending the generators of the 2 copies of $\mathbb{Z}_2$ to multiplication by $(-1, 1)$ and $(1, -1)$, respectively. Then if S^3 is the unit sphere in $(\mathbb{C}^2)^\wedge$, $S^3 \times S^3$ is a closed Q-invariant subset of $(\mathbb{C}^4)^\wedge$ and so gives rise to a quotient of $C^*(G)$ isomorphic to $C(S^3 \times S^3) \rtimes Q$. This crossed product is a direct sum of two ideals (corresponding to the two possible actions of $Z(Q)$ in an irreducible representation), namely $C(S^3 \times S^3) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $C^*(C(S^3 \times S^3), \mathbb{Z}_2 \times \mathbb{Z}_2, \omega)$, where the latter is the twisted crossed product associated to the cocycle $\omega \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ defining the group extension $1 \to Z(Q) \to Q \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$. As shown in [**35**], the Dixmier-Douady invariant of the twisted crossed product is a nonzero torsion-class.

EXAMPLE 4.20. Here is a simple example of a connected, simply connected (nonexponential) solvable Lie group G whose group C^*-algebra contains an essential continuous-trace ideal with $\delta \neq 0$. Merely let $G = \mathbb{C}^2 \rtimes_\beta \mathbb{R}$, where $\beta_t(w, z) = (e^{2\pi i t} w, e^{2\pi i t} z)$. This is a solvable Lie group of dimension 5, and from the β-equivariant decomposition $\hat{\mathbb{C}}^2 = \{0\} \cup (0, \infty) \times S^3$, one sees that $C^*(G)$ is an extension

$$0 \to C_0((0, \infty)) \otimes \left(C(S^3) \rtimes_\beta \mathbb{R}\right) \to C^*(G) \to C_0(\mathbb{R}) \to 0.$$

By Example 4.6, $C(S^3) \rtimes_\beta \mathbb{R}$ is a continuous-trace algebra with spectrum $S^1 \times S^2$ and Dixmier-Douady class a generator of H^3, so that if $I = C_0(\mathbb{R}) \otimes (C(S^3) \rtimes \mathbb{R})$, I is a continuous-trace ideal of $C^*(G)$ with spectrum $(0, \infty) \times S^1 \times S^2$ and Dixmier-Douady class a generator of H^3. Incidentally, the extension cannot split even stably, since by the Thom isomorphism theorem of [**3**] or by [**16**, §16], $K_1(C^*(G)) \cong \mathbb{Z}$ and $K_0(C^*(G)) = 0$, whereas by Example 4.6, $K_0(I) \cong K_1(I) \cong \mathbb{Z}$. Thus the connecting map $K_1(C_0(\mathbb{R})) \to K_0(I)$ must be an isomorphism. (Compare the similar analyses in [**30**, §7].) □

In spite of all this, there seems to be definite evidence for Conjecture 4.18. If G is exponential and we write as before $G = N \rtimes \mathbb{R}$, where N is a subgroup of codimension 1, then by [**2**, Chapter I, 3.3], every point in $\hat{N}$ has stability group $\{0\}$ or $\mathbb{R}$. (Unfortunately, however, the action of $\mathbb{R}$ on the complement of the fixed points in $\hat{N}$ need not be proper.) And as we have seen in the proof of Theorem 4.14, if $\mathbb{R}$ acts on a continuous-trace algebra B with $\delta(B) = 0$, and if the action on $\hat{B}$ is either trivial or free and proper, then the crossed product $A = B \rtimes \mathbb{R}$ will again be a continuous-trace algebra with $\delta(A) = 0$. This suggests that Conjecture 4.18 could be proven by induction on $\dim G$, provided one could control behavior at places where the orbit type of $\mathbb{R}$ on $\hat{N}$ changes. However, this seems difficult to do, so probably Conjecture 4.18 is about as difficult as the problem about bicontinuity of the Kirillov-Bernat bijection.

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